A DETERMINATION OF THE ORBIT OF THE ASTEROID 522 HELGA

by

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TO MY PARENTS

ΝΙΚΟΛΑΟΣ - ΕΤΤΤΧΙΑ

TO MY SUPERVISOR

TADEUSZ

Εἰ πρῶτον μή εὐτυχήσω, αὖθις αὖ πάλιν πειρᾶσαι.

PLATO

DECLARATION

I hereby declare that the following Thesis is the result of work carried out by me, that the Thesis is my own composition, and that it has not previously been presented for a Higher Degree. The research was carried out at the University Observatory, St Andrews.

D. N. Papadakos

CERTIFICATE

I hereby certify that Dimitrios N. Papadakos has spent seven terms at research work in the University Observatory, St Andrews, that he has fulfilled the conditions of Ordinance No. 51 (St Andrews) and that he is qualified to submit the accompanying Thesis in application for the Degree of Master of Science.

T. B. Slebarski

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CONTENTS

INTRODUCTION

There are two principal themes of this work.

Firstly, the application of the Gaussian and Laplacian methods to determine the preliminary orbit of the minor planet 522 Helga.

Secondly, the application of the power series method to find:

(i) The velocity vectors of the planets of the Solar System at a given time.

(ii) The determination of the real paths of these planets and hence the path of the minor planet 522 Helga.

By applying the latter method we are able to determine the mass of the minor planet (1.10^{-13} solar masses) and to construct its ephemeris.

The calculations were carried out by using computer and all programs are available.

The photographic plates used in this work were taken with the James Gregory Telescope at the University Observatory (St Andrews) during the observing season of 1971-72.

Chapter 1
Gaussian Method

The solution of the problem of the relative motion of two bodies, which can be regarded as particles, contains six arbitrary constants of integration. They are:

h_x, h_y, h_z - components of angular momentum in a given rectangular system of reference.

C - energy constant.

ω - orientation of the major axis of the orbit

T - time when the bodies are at the least distance from each other.

In problems related to planets and comets it is more convenient to use six elements, which are independent functions of the constants of integration. They are:

e - eccentricity of the orbit.

a - semi-major axis of the orbit.

M_0 - mean anomaly of a planet at a chosen epoch t_0

ω - argument of perihelion (orientation of the major axis).

i - inclination of the orbit.

Ω - longitude of ascendinal node.

Of course, the choice of the above orbital elements is not unique. It is possible to replace this set by another.

The reduction of a set of observed topocentric co-ordinates α_i, δ_i $(i = 1, 2, 3)$ of a minor planet to obtain its orbital elements consists the problem of the initial determination of the orbit of the minor planet. Such a solution gives a satisfactory representation of the real motion during a relatively short interval of time, as long as it is possible to neglect the action of the other planets. Therefore, the observations must be separated by short intervals of time.

Let $\underset{\sim}{r_i} \equiv (r_{x_i}, r_{y_i}, r_{z_i})$ $(i = 1,2,3)$ be the heliocentric radius vectors of the minor planet at three instants t_i $(i = 1,2,3)$. Similarly, let $\underset{\sim}{R_i} \equiv (R_{x_i}, R_{y_i}, R_{z_i})$ and $\underset{\sim}{\rho_i} \equiv (\rho_{x_i}, \rho_{y_i}, \rho_{z_i})$ be the topocentric radius vectors of the Sun and the minor planet respectively at the same instants. Since the orbit in Keplerian mechanics, neglecting the action of the other planets, is a plane passing through the origin of the heliocentric co-ordinates we can write:

$$\underset{\sim}{r_2} = C_1 \underset{\sim}{r_1} + C_3 \underset{\sim}{r_3} , \tag{1.1}$$

which states that the three position vectors $\underset{\sim}{r_1}, \underset{\sim}{r_2}, \underset{\sim}{r_3}$ are dependent, so the position vector $\underset{\sim}{r_2}$, at any time t_2, is some linear combination of the radius vectors at t_1 and t_3. The equation (1.1) gives

$$(\underset{\sim}{r_2} \times \underset{\sim}{r_3}) = C_1 (\underset{\sim}{r_1} \times \underset{\sim}{r_3}) , \quad (\underset{\sim}{r_1} \times \underset{\sim}{r_2}) = C_3 (\underset{\sim}{r_1} \times \underset{\sim}{r_3}) .$$

Hence

$$C_1 = \frac{[r_2, r_3]}{[r_1, r_3]} , \qquad C_3 = \frac{[r_1, r_2]}{[r_1, r_3]} , \tag{1.2}$$

where, in general, $[r_i, r_j]$ $(i, j = 1\ 2\ 3 \quad i \neq j)$ stands for the area of the triangle formed by $\underset{\sim}{r_i}$ and $\underset{\sim}{r_j}$. C's are known as the "triangle ratios".

The vectors $\underset{\sim}{r_i}, \underset{\sim}{R_i}, \underset{\sim}{\rho_i}$ defined above, are related by

$$\underset{\sim}{r_i} = \underset{\sim}{\rho_i} - \underset{\sim}{R_i} \quad (i = 1,2,3) . \tag{1.3}$$

Combining the expressions (1.1) and (1.3) we obtain

$$C_1 \underset{\sim}{\rho_1} - \underset{\sim}{\rho_2} + C_3 \underset{\sim}{\rho_3} = C_1 \underset{\sim}{R_1} - \underset{\sim}{R_2} + C_3 \underset{\sim}{R_3} , \tag{1.4}$$

which is one of the fundamental equations of the Gaussian method.

Operating upon both sides of the equation (1.4) by $\cdot(\underset{\sim}{\hat{\rho}_1} \times \underset{\sim}{\hat{\rho}_3})$, where $\underset{\sim}{\hat{\rho}_i} = \underset{\sim}{\rho_i} / |\underset{\sim}{\rho_i}|$ $(i = 1,2,3)$, we get

$$-\rho_2 \underset{\sim}{\hat{\rho}_2} \cdot (\underset{\sim}{\hat{\rho}_1} \times \underset{\sim}{\hat{\rho}_3}) = C_1 \underset{\sim}{R_1} \cdot (\underset{\sim}{\hat{\rho}_1} \times \underset{\sim}{\hat{\rho}_3}) - \underset{\sim}{R_2} \cdot (\underset{\sim}{\hat{\rho}_1} \times \underset{\sim}{\hat{\rho}_3}) + C_3 \underset{\sim}{R_3} \cdot (\underset{\sim}{\hat{\rho}_1} \times \underset{\sim}{\hat{\rho}_3}) . \tag{1.5}$$

Since the $\underset{\sim}{\hat{\rho}_i} = (\cos\delta_i \cos\alpha_i, \cos\delta_i \sin\alpha_i, \sin\delta_i)$ and $\underset{\sim}{R_i}$ $(i = 1,2,3)$ are known, the equation (1.5) reduces to an equation of the form

$$W_1 \rho_2 = W_2 C_1 + W_3 + W_4 C_3 \quad . \tag{1.6}$$

Consider now the triangle formed by the sun, the observer and the planet at t_2 . This has sides of length r_2, R_2 and ρ_2 and thus, we can write

$$r_2^2 = \rho_2^2 - 2\rho_2 \hat{\underset{\sim}{\rho}}_2 \cdot \underset{\sim}{R}_2 + R_2^2 \quad , \tag{1.7}$$

which, since $\hat{\underset{\sim}{\rho}}_2$ and $\underset{\sim}{R}_2$ are known, is an equation of the form

$$r_2^2 = \rho_2^2 + W_{11}\rho_2 + W_{12} \quad . \tag{1.8}$$

The first step in the computation is to calculate the coefficients $W_1, W_2, W_3, W_4, W_{11}$ and W_{12} using the topocentric direction cosines of the planet, obtained from its observed topocentric co-ordinates α_i, δ_i $(i=1,2,3)$, and the topocentric rectangular co-ordinates, obtained from the geocentric tables, corrected for parallax and reduced to the place of observations by introduction of appropriate corrections.

We can now proceed to obtain first approximations to the r_i $(i=1,2,3)$. In order to do this we require to know C_1 and C_3 . From the expansions of the co-ordinates of the body in convergent Taylor's series we get

$$C_1 = \frac{\tau_1}{\tau_2}\left[1 + \frac{(\tau_2^2 - \tau_1^2)}{6r_2^3} + \frac{\tau_3(\tau_2\tau_3 - \tau_1^2)}{4r_2^4}\left(\frac{dr}{d\tau}\right)_{t=t_2} + \cdots\right] ,$$

and

$$C_3 = \frac{\tau_3}{\tau_2}\left[1 + \frac{(\tau_2^2 - \tau_3^2)}{6r_2^3} + \frac{\tau_1(\tau_2\tau_1 - \tau_3^2)}{4r_2^4}\left(\frac{dr}{d\tau}\right)_{t=t_2} + \cdots\right] , \tag{1.9}$$

where $\tau_1 = k\cdot(t_3 - t_2)$, $\tau_2 = k\cdot(t_3 - t_1)$, $\tau_3 = k\cdot(t_2 - t_1)$ are the modified times, and k is the Gaussian constant. Since terms with $\left(\frac{dr}{d\tau}\right)_{t=t_2}$ are unknown, in the first approximation we have to neglect them. Thus we shall set

$$C_1 = \frac{\tau_1}{\tau_2}\left[1 + \frac{\tau_2^2 - \tau_1^2}{6r_2^3}\right] , \quad C_3 = \frac{\tau_3}{\tau_2}\left[1 + \frac{\tau_2^2 - \tau_3^2}{6r_2^3}\right] \quad . \tag{1.10}$$

Clearly, since τ_1, τ_2 and τ_3 are known, the elimination of C_1 and C_3 in equation (1.6) by substitution of the above expressions (1.10) will result in an equation of the form

$$\rho_2 = W_9 + W_{10} r_2^{-3} \quad . \tag{1.11}$$

It is obvious that we can obtain values for ρ_2 and r_2 by simultaneous solution of the equations (1.11) and (1.8). This is accomplished by means of successive approximations. We can take $r_2 = 2.8$ A.U. as first approximation since this is the mean distance of the minor planets from the Sun. This value is used in equation (1.11) to obtain a value of ρ_2 which is then inserted into equation (1.8) to obtain a second approximation of r_2. The obtained value of r_2 is substituted into equation (1.11) and the process is continued until successive approximations of ρ_2 and r_2 are identical to the required accuracy.

Since equations (1.10) are nothing more than preliminary estimates of the triangle ratios C_1 and C_3, the r_2 and ρ_2 obtained by the solution of the equations (1.8) and (1.11) are only approximate. It should be noted that the coefficients W_{11} and W_{12} in equation (1.8) are exact and they do not change in the subsequent computation of $\underset{\sim}{r_i}$ $(i = 1, 2, 3)$.

The next step in the reduction is to use the value of r_2 to obtain first approximations for C_1 and C_3 by means of the expressions (1.10). After these unknowns are determined, we may return to the fundamental equation (1.4) to obtain ρ_1 and ρ_3. The equation is used in the form

$$\begin{aligned} C_1 |\hat{\rho}_{x1}| \rho_1 + C_3 |\hat{\rho}_{x3}| \rho_3 &= C_1 R_{x1} - R_{x2} + C_3 R_{x3} + |\hat{\rho}_{x2}| \rho_2 \,, \\ C_1 |\hat{\underset{\sim}{\rho}}_{y1}| \rho_1 + C_3 |\hat{\underset{\sim}{\rho}}_{y3}| \rho_3 &= C_1 R_{y1} - R_{y2} + C_3 R_{y3} + |\hat{\underset{\sim}{\rho}}_{y2}| \rho_2 \,, \\ C_1 |\hat{\underset{\sim}{\rho}}_{z1}| \rho_1 + C_3 |\hat{\underset{\sim}{\rho}}_{z3}| \rho_3 &= C_1 R_{z1} - R_{z2} + C_3 R_{z3} + |\hat{\underset{\sim}{\rho}}_{z2}| \rho_2 \,, \end{aligned} \tag{1.12}$$

where $|\hat{\underset{\sim}{\rho}}_{xi}| = \cos\delta_i \cos\alpha_i$, $|\hat{\underset{\sim}{\rho}}_{yi}| = \cos\delta_i \sin\alpha_i$ and $|\hat{\underset{\sim}{\rho}}_{zi}| = \sin\delta_i$ $(i = 1, 2, 3)$.

The expressions (1.12) contain two unknowns, ρ_1 and ρ_3, therefore we require two of these component equations to obtain a solution. The third equation is used to provide a numerical check on the values of ρ_1 and ρ_3.

This is generally unnecessary when a computer is used for the reduction.

Our observations are all exactly satisfied since our values of the ρ_1, ρ_2 and ρ_3 have been derived directly from the equations which express the geometrical conditions. Our C's however, were obtained from approximate formulas and therefore there is no guarantee that the motion of the minor planet is in strict accordance with the law of gravitation.

With the ρ's we now have, we may obtain the corresponding $\underset{\sim}{r}$'s by means of the equation

$$\underset{\sim}{r_i} = \rho_i \underset{\sim}{\hat{\rho}_i} - \underset{\sim}{R_i} \quad (i = 1,2,3)\,. \tag{1.13}$$

In the second approximation, knowing ρ_1, ρ_2 and ρ_3, we are able to take into consideration the neglected influence of planetary aberration on the moments of observations. The corrected moments will be

$$\overset{\circ}{t}_i = t_i^* - A\rho_i \quad (i = 1,2,3)\,. \tag{1.14}$$

It is necessary to substitute them in place of the initial t_i in the calculation of the formulas. $\overset{\circ}{t}_i$ is the moment when the light ray observed by us left the celestial body, t_i^* is the ephemeris time of observation and A is the light time for unit distance (time during which light traves one astronomical unit). To evaluate t_i^* we use the formula

$$t_i^* = t_i + \Delta T \quad (i = 1,2,3)\,, \tag{1.15}$$

where t_i is the moment of observation indicated in U.T. and ΔT the correction to U.T. to obtain the ephemeris time. The equation (1.14) by introduction of the equation (1.15) becomes

$$\overset{\circ}{t}_i = t_i + \Delta T - A\rho_i \quad (i = 1,2,3)\,. \tag{1.16}$$

More accurate expressions for C_1 and C_3 was given by Gibbs. They are

$$C_1 = \frac{\tau_1}{\tau_2} \cdot \frac{1 + B_1 r_1^{-3}}{1 - B_2 r_2^{-3}}\,, \qquad C_3 = \frac{\tau_3}{\tau_2}\;\frac{1 + B_3 r_3^{-3}}{1 - B_2 r_2^{-3}}\,, \tag{1.17}$$

where $B_1=(\tau_3^2+\tau_1\tau_3-\tau_1^2)/12$, $B_2=(\tau_1^2+3\tau_1\tau_3+\tau_3^2)/12$, $B_3=(\tau_1^2+\tau_1\tau_3-\tau_3^2)/12$

and $\quad \tau_1=k\,(t_3^\circ-t_2^\circ)$, $\tau_2=k\,(t_3^\circ-t_1^\circ)$, $\tau_3=k\,(t_2^\circ-t_1^\circ)$.

Taking into account only the first two terms of the series expansions (1.9) for C's, we have

$$C_1=\frac{\tau_1}{\tau_2}+\frac{\nu_1}{r_2^3} , \quad C_3=\frac{\tau_3}{\tau_2}+\frac{\nu_3}{r_2^3} , \tag{1.18}$$

where

$$\nu_1=\frac{\tau_1}{\tau_2}\,\frac{\tau_2^2-\tau_1^2}{6} , \quad \nu_3=\frac{\tau_3}{\tau_2}\,\frac{\tau_2^2-\tau_3^2}{6} .$$

The equations (1.18) give

$$\nu_1=\left(C_1-\frac{\tau_1}{\tau_2}\right)r_2^3 , \quad \nu_3=\left(C_3-\frac{\tau_3}{\tau_2}\right)r_2^3 . \tag{1.19}$$

Using the found value of r_2 from the previous approximation and the values of C_1 and C_3 evaluated from the Gibbs formulaé (1.17), we compute the values of ν's. Substitution of these equations into equations (1.18) gives

$$C_1=W_5+W_6\,r_2^{-3} , \quad C_3=W_7+W_8\,r_2^{-3} , \tag{1.20}$$

where $W_5=\tau_1/\tau_2$, $W_6=\nu_1$, $W_7=\tau_3/\tau_2$, $W_8=\nu_3$ are known. The equation (1.6) by introduction of the equations (1.20) takes the form

$$\rho_2=W_9+W_{10}\,r_2^{-3} . \tag{1.21}$$

The equations (1.8) and (1.21) are now solved for r_2 and ρ_2 .

The value of r_2 obtained in the previous approximation, is taken as the initial value of the iteration. The values of C_1 and C_3 are determined using (1.20), and the new values of ρ_1 and ρ_3 are obtained from the equations (1.12). Then the vectors $\underset{\sim}{r_i}$ $(i=1,2,3)$ are determined by means of

$$\underset{\sim}{r}_i = \rho_i \, \underset{\sim}{\hat{\rho}}_i - \underset{\sim}{R}_i \qquad (i=1,2,3) \ .$$

This process must be repeated by exactly the same method until the values of C's and $\underset{\sim}{r}$'s no longer change. The convergence is, in general, very rapid. This is due to the fact that C's are relatively insensitive to the $\underset{\sim}{r}$'s. The large change in $\underset{\sim}{r}$'s, in going from one approximation to the next, will not produce a correspondingly large change in C's. Hence the next solution for the ρ's will not differ greatly from the preceding one. As a rule, no more than four approximations lead to the final values of $\underset{\sim}{r}_i$ $(i=1,2,3)$.

The next stage in the computation is the determination of the ratio of the area of the sector to that of the triangle in the orbit. This important concept is fundamental to the Gaussian method and its determination forms an intermediate stage between the calculation of the $\underset{\sim}{r}_i$ $(i=1,2,3)$ and the evaluation of the orbital elements. It is possible to obtain the ratio of the area of the sector to the area of the triangle from two heliocentric positions of the body and the interval of time during which the body passed from the first position to the second.

Let r_i, r_j be the heliocentric distances of the planet at two instants t_i, t_j respectively, and E_i, E_j the corresponding eccentric anomalies. Also let $t_j > t_i$.

The "sector-triangle ratio", $\tilde{y}_k$, satisfies the Gauss equations

$$\tilde{y}_k^2 = \frac{m_k^2}{\ell_k + \sin^2(g_k/2)} , \quad \tilde{y}_k^3 - \tilde{y}_k^2 = \frac{m_k^2(2g_k - \sin 2g_k)}{\sin^3 g_k} , \tag{1.22}$$

where

$$m_k^2 = \frac{\tau_k^2}{2\sqrt{2}\,K_k^3} , \quad \ell_k = \frac{r_i + r_j}{2\sqrt{2}\,K_k} - \frac{1}{2} ,$$

$$K_k^2 = r_i r_j + \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j , \qquad 2g_k = E_j - E_i , \quad \tau_k = k(t_j - t_i) \ .$$

The equations (1.22) contain two unknowns $\tilde{y}_k$ and g_k. Since it is not possible to eliminate the later, one has to use a method of successive approximations to obtain $\tilde{y}_k$. In the determination of a preliminary orbit, the difference in eccentric anomalies is usually a small quantity. Then, using expansion in series, Gauss derived the equation

$$\tilde{y}_k^3 - \tilde{y}_k^2 - h_k \tilde{y}_k - \frac{1}{9} h_k = 0 , \tag{1.23}$$

where

$$h_k = \frac{m_k^2}{5/6 + \ell_k + \xi_k}, \qquad \xi_k = \frac{2}{35} x_k^2 + \frac{52}{1575} x_k^3 + \cdots , \tag{1.24}$$

$$x_k = \frac{m_k^2}{\tilde{y}_k^2} - \ell_k = \sin^2(g_k/2) .$$

The cubic equation (1.23) has only one positive root, which, for small values of h_k, can be obtained using Hansen's approximate formula

$$\tilde{y}_k = 1 + \cfrac{\frac{10}{9} h_k}{1 + \cfrac{11/9\, h_k}{1 + \cfrac{11/9\, h_k}{1 + \cdots}}} = 1 + \frac{10}{11} \cfrac{\frac{11}{9} h_k}{1 + \cfrac{\frac{11}{9} h_k}{1 + \cfrac{\frac{11}{9} h_k}{1 + \cdots}}} . \tag{1.25}$$

The procedure for determining $\tilde{y}_i$ $(i=1,2,3)$ for the three observations is as follows: firstly, using the final values of the $\underset{\sim}{r}_i$ $(i=1,2,3)$ obtained in the previous section of the work, we calculate the K_i $(i=1,2,3)$

$$\begin{aligned} K_1^2 &= r_2 r_3 + \underset{\sim}{r}_2 \cdot \underset{\sim}{r}_3 , \\ K_2^2 &= r_1 r_3 + \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_3 , \\ K_3^2 &= r_2 r_1 + \underset{\sim}{r}_2 \cdot \underset{\sim}{r}_1 . \end{aligned} \tag{1.26}$$

We next compute the m_i^2 and l_i $(i=1,2,3)$ using the equations (1.22). That is

$$m_i^2 = \frac{\tau_i^2}{2\sqrt{2}\,K_i^3} \quad , \qquad l_i = \frac{r_k + r_j}{2\sqrt{2}\,K_i} - \frac{1}{2} \quad (i,j,k = 1,2,3 \quad i \neq j \neq k),$$

where τ_i $(i=1,2,3)$ are the final values, corrected for light time, obtained in the previous section.

The first approximations for the h_i $(i=1,2,3)$ are obtained by neglecting the ξ_i $(i=1,2,3)$

$$h_i = \frac{m_i^2}{5/6 + l_i} \quad (i=1,2,3).$$

These values are substituted in the equation (1.25) to obtain the first approximations to the $\tilde{y}_i$ $(i=1,2,3)$. These are then used to compute x_i and ξ_i from (1.24).

Now, the second approximation to h_i (i = 1, 2, 3) is determined with the help of the first of the equations (1.24). Second approximation to $\tilde{y}_i$ $(i=1,2,3)$ is then derived, and the process is repeated until the successive values of the h_i and $\tilde{y}_i$ $(i=1,2,3)$ agree sufficiently well.

By definition

$$\tilde{y}_k = \frac{(r_i, r_j)}{[r_i, r_j]} \qquad i,j,k = 1,2,3 \quad i \neq j \neq k, \tag{1.27}$$

where (r_i, r_j) represents the area of the sector of the ellipse contained between the two radius vectors $\underset{\sim}{r_i}$ and $\underset{\sim}{r_j}$. With the help of the equations (1.27) and (1.2) we easily obtain

$$C_1 = \frac{(r_2, r_3)\,\tilde{y}_2}{(r_1, r_3)\,\tilde{y}_1} \quad , \qquad C_3 = \frac{(r_1, r_2)\,\tilde{y}_2}{(r_1, r_3)\,\tilde{y}_3}.$$

From the Kepler's laws it follows that the areas of the sectors are proportional to the intervals of time between observations. Hence

$$C_1 = \frac{\tau_1\,\tilde{y}_2}{\tau_2\,\tilde{y}_1} \quad , \qquad C_3 = \frac{\tau_3\,\tilde{y}_2}{\tau_2\,\tilde{y}_3}.$$

We use these equations as a check on the values obtained for $\tilde{y}_i$ $(i=1,2,3)$ The values of C's obtained from these expressions are compared with the values obtained in the final approximation to the $\underset{\sim}{r_i}$ $(i=1,2,3)$. There should be good agreement between the two sets of values if the accuracy has been maintained.

We now have values of $\underset{\sim}{r_i}$, τ_i and y_i $(i=1,2,3)$ and can thus proceed to the computation of the orbital elements.

We notice that, in the numerical applications, we shall use dimensionless quantities. Therefore, the following units of time, mass and distance will be employed. The mass of the Sun will be taken as the unit of mass, the astronomical unit as the unit of distance, and $1/k$ mean solar days as the unit of time.

Considering the mass of the asteroid negligible in comparison with the mass of the Sun, the "sector-triangle ratios" $\tilde{y}_1, \tilde{y}_2$ and $\tilde{y}_3$ are given by

$$\tilde{y}_1 = \frac{\tau_1 \sqrt{P}}{r_2 r_3 \sin(v_3 - v_2)}, \quad \tilde{y}_2 = \frac{\tau_2 \sqrt{P}}{r_1 r_3 \sin(v_3 - v_1)}, \quad \tilde{y}_3 = \frac{\tau_3 \sqrt{P}}{r_2 r_1 \sin(v_2 - v_1)},$$

where v_1, v_2 and v_3 are the true anomalies of the minor planet at the instants t_1, t_2 and t_3 respectively. From these equations we obtain the parameter of the orbit P as

$$P = \frac{(r_1 r_3)^2 \sin^2(v_3 - v_1)}{\tau_2^2} \tilde{y}_2^2 = \frac{(r_2 r_3)^2 \sin^2(v_3 - v_2)}{\tau_1^2} \tilde{y}_1^2 = \frac{(r_1 r_2)^2 \sin^2(v_2 - v_1)}{\tau_3^2} \tilde{y}_3^2 .$$

In order to determine P, it is sufficient to make use of the first of these equations, which gives the greatest accuracy, because its numerator and denominator are larger than those in the two following equations. In practice we proceed as follows: Let us set

$$\sigma = \frac{\underset{\sim}{r_1} \cdot \underset{\sim}{r_3}}{r_1^2} = \frac{r_{x1} r_{x3} + r_{y1} r_{y3} + r_{z1} r_{z3}}{r_1^2} = \frac{r_3 \cos(v_3 - v_1)}{r_1},$$

and let us introduce the notation

$$\underset{\sim}{r_0} = \underset{\sim}{r_3} - \sigma \underset{\sim}{r_1} \, .$$

It follows that

$$r_0^2 = r_3^2 - \sigma^2 r_1^2 = r_3^2 \sin^2(v_3 - v_1),$$

and

$$r_1 r_3 \sin(v_3 - v_1) = r_1 r_0 \, .$$

Hence

$$p = \frac{(r_1 r_0)^2}{\tau_2^2} \tilde{y}_2^2 \, . \tag{1.28}$$

For the determination of the true anomalies we use the relations

$$r_1 = \frac{p}{1 + e\cos v_1}, \qquad r_3 = \frac{p}{1 + e\cos v_3},$$

which give

$$e\cos v_1 = \frac{p}{r_1} - 1, \qquad e\cos v_3 = \frac{p}{r_3} - 1 \, . \tag{1.29}$$

The last equation can be represented in the form

$$e\cos v_3 = e\cos\big(v_1 + (v_3 - v_1)\big) = e\cos v_1 \cos(v_3 - v_1) - e\sin v_1 \sin(v_3 - v_1) = \frac{p}{r_3} - 1 \, .$$

Whence

$$e\sin v_1 = \frac{\left(\frac{p}{r_1} - 1\right)\cos(v_3 - v_1) - \left(\frac{p}{r_3} - 1\right)}{\sin(v_3 - v_1)} \, . \tag{1.30}$$

Evaluating

$$\cos(v_3 - v_1) = \frac{\underset{\sim}{r_1} \cdot \underset{\sim}{r_3}}{r_1 r_3},$$

$$\sin(v_3 - v_1) = \left(1 - \cos^2(v_3 - v_1)\right)^{\frac{1}{2}}, \tag{1.31}$$

the equations (1.29) and (1.30) permit us to find v_1 and e.

With the help of the relations (1.31)

$$v_3 = v_1 + (v_3 - v_1)$$

is also obtained.

The semi-major axis a and the mean daily motion n are directly obtained from

$$a = p\,(1-e^2)^{-1}\,, \quad n = k\,a^{-3/2}\,.$$

For elliptical orbits we find the eccentric anomalies by the formulas

$$\tan\frac{E_1}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{v_1}{2}\,, \quad \tan\frac{E_3}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{v_3}{2}\,,$$

and then we apply Kepler's equation giving the mean anomaly

$$M_i = E_i - e\sin E_i \qquad (i=1,3)\,.$$

The time of perihelion passage is computed by use of the formula

$$T = t_i^{\circ} - \frac{M_i}{n} \qquad (i=1,3)\,,$$

where t_i° $(i=1,3)$ is the E.T. corrected for planetary aberration.

The element which normally replaces T in elliptical orbits is the mean anomaly M_0 at some specific epoch t_0. M_0 can be calculated from

$$M_0 = n\,(t_0 - T) \quad \text{or} \quad M_0 = M_i + n\,(t_0 - t_i) \quad (i=1,3)\,,$$

where, again, t_i $(i=1,3)$ is the E.T. corrected for planetary aberration.

To find the remaining three elements ω, i and Ω we require a knowledge of the vectorial equatorial constants $P_x, P_y, P_z, Q_x, Q_y, Q_z$ which are components, in the equatorial system, of the unit vectors $\hat{\underset{\sim}{P}}$ and $\hat{\underset{\sim}{Q}}$ directed along the major and minor axis of the orbit respectively.

The equations

$$\mathbf{r}_1 = \hat{\mathbf{P}}\, r_1 \cos U_1 + \hat{\mathbf{Q}}\, r_1 \sin U_1 ,$$
$$\mathbf{r}_3 = \hat{\mathbf{P}}\, r_3 \cos U_3 + \hat{\mathbf{Q}}\, r_3 \sin U_3 ,$$

give us

$$\hat{\mathbf{P}} = \frac{\mathbf{r}_1 r_3 \sin U_3 - \mathbf{r}_3 r_1 \sin U_1}{r_1 r_3 \sin(U_3 - U_1)} ,$$

$$\hat{\mathbf{Q}} = \frac{\mathbf{r}_3 r_1 \cos U_1 - \mathbf{r}_1 r_3 \cos U_3}{r_1 r_3 \sin(U_3 - U_1)} ,$$

from which $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ could be evaluated directly, but it is more convenient to proceed using the already defined vector $\mathbf{r}_0$. Whence

$$\hat{\mathbf{P}} = \frac{\cos U_1}{r_1}\mathbf{r}_1 - \frac{\sin U_1}{r_0}\mathbf{r}_0 ,$$

$$\hat{\mathbf{Q}} \quad \frac{\sin U_1}{r_1}\mathbf{r}_1 - \frac{\cos U_1}{r_0}\mathbf{r}_0 .$$

The elements i, Ω and ω are referred to the ecliptic, therefore we have to rotate the co-ordinate system about X-axis through an angle ε equal to the mean obliquity of 1950.0. If the ecliptic vectorial constants are $P'_x, P'_y, P'_z, Q'_x, Q'_y, Q'_z, R'_x, R'_y, R'_z$, then

$$\begin{bmatrix} P'_x & P'_y & P'_z \\ Q'_x & Q'_y & Q'_z \\ R'_x & R'_y & R'_z \end{bmatrix} \quad \begin{bmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} .$$

Alternatively,

$$\begin{aligned} \sin\omega \sin i &= P_z \cos\varepsilon - P_y \sin\varepsilon , \\ \cos\omega \sin i &= Q_z \cos\varepsilon - Q_y \sin\varepsilon , \\ \sin\Omega &= (P_y \cos\omega - Q_y \sin\omega)\sec\varepsilon , \\ \cos\Omega &= P_x \cos\omega - Q_x \sin\omega , \\ \cos i &= -(P_x \sin\omega + Q_x \cos\omega)\operatorname{cosec}\Omega . \end{aligned}$$

From these equations we obtain ω, i and Ω . In the numerical applications we must take care that the found values of the angles are the values of the corresponding orbital elements, since the same value of trigonometric function corresponds to more than one angle.

Chapter 2
Laplacian method

The problem of the determination of elements by Laplace's method can, briefly, be described as follows: From the known topocentric position vectors of the Sun $\underset{\sim}{R}_i$ $(i=1,2,3)$ at the instants t_1, t_2, t_3 and the unit vectors $\underset{\sim}{\hat{\rho}}_i$ $(i=1,2,3)$ along the topocentric position vectors of the planet, we determine its heliocentric position and velocity vectors at the instant $t = t_2$ in the rectangular ecliptic system of co-ordinates, and from these we find the elements of orbit.

Before we continue with the method we shall explain how we can determine the velocity vector of a planet at the instant t_0, given its two position vectors at t_1 and t_2 and the position vector at t_0, where $t_1 < t_0 < t_2$.

If the attraction of the other bodies are neglected, the acceleration of the planet relative to the Sun is given by

$$\ddot{\underset{\sim}{r}} = -k^2(1+m)\frac{\underset{\sim}{r}}{r^3}, \tag{2.1}$$

where k is the Gaussian constant and m the mass of the planet expressed in terms of the solar mass.

We define τ by

$$\tau = k(t-t_0),$$

and introduce the following notation

$$\underset{\sim}{r}' = \frac{d\underset{\sim}{r}}{d\tau}, \quad \underset{\sim}{r}'' = \frac{d^2\underset{\sim}{r}}{d\tau^2}, \quad \underset{\sim}{r}^{(\nu)} = \frac{d^\nu\underset{\sim}{r}}{d\tau^\nu}, \quad \nu = 3, 4 \cdots\cdot$$

The equation (2.1) becomes

$$\underset{\sim}{r}'' = -(1+m)\frac{\underset{\sim}{r}}{r^3}. \tag{2.2}$$

We write the solution of (2.2) in a Taylor's series

$$\underset{\sim}{r} = \underset{\sim}{r_o} + \underset{\sim}{r_o}'\tau + \underset{\sim}{r_o}''\frac{\tau^2}{2!} + \sum_{\nu=3}^{\infty} \underset{\sim}{r_o}^{(\nu)}\frac{\tau^\nu}{\nu!} , \qquad (2.3)$$

where $\underset{\sim}{r_o}$ is the position vector at t=to, and the subscript o refers to the values for this position.

The differential equation (2.2) permits us to eliminate $\underset{\sim}{r_o}''$ in terms of $\underset{\sim}{r_o}$ and $r_o = |\underset{\sim}{r_o}|$. Similarly, the higher derivatives in the equation (2.3) can be eliminated by successive differentiation of the equation (2.2). This process leads to an expression

$$\underset{\sim}{r} = f\,\underset{\sim}{r_o} + g\,\underset{\sim}{r_o}' , \qquad (2.4)$$

where $f = \sum_{n=0}^{\infty} f_n \frac{\tau^n}{n!}$ and $g = \sum_{n=0}^{\infty} g_n \frac{\tau^n}{n!}$.

These power series in τ are known as "f and g series".

The coefficients f_n and g_n are polynomials in μ_o, σ_o and ε_o, defined by

$$\mu_o = \frac{(1+m)}{r_o^3} , \quad \sigma_o = \frac{r_o'}{r_o} , \quad \varepsilon_o = \frac{r_o'^2}{r_o^2} - \mu_o . \qquad (2.5)$$

Lagrange was first to give explicit expressions for f_n and g_n up to $n = 5$. The direct derivations for $n > 5$ becomes very laborious. Cipoletti (1872) discovered two recursive formulae

$$\begin{aligned} f_n &= f_{n-1}' - (1/r_o^3)\, g_{n-1} , \\ g_n &= f_{n-1} - g_{n-1}' , \end{aligned} \qquad (2.6)$$

with initial values $f_0 = 1$ and $g_0 = 0$. The tedious algebra involved in the evaluation of the formula (2.6) can be carried out by a computer. Sconzo (1965) obtained f_n and g_n on an IBM 7094 computer, up to n = 27.

In the present work the coefficients f_n and g_n up to 12$^{\text{th}}$ order will be used. The expressions for these are given in the table I.

Consider now two position vectors $\underset{\sim}{r_1}$, $\underset{\sim}{r_2}$ at the instants τ_1, τ_2, respectively. Using the power series expansion (2.3) about the same epoch, approximately halfway between τ_1 and τ_2, we obtain

$$\underset{\sim}{r_1} = \underset{\sim}{r_0} + \underset{\sim}{r_0}'\tau_1 + \underset{\sim}{r_0}''\frac{\tau_1^2}{2!} + \underset{\sim}{r_0}^{(3)}\frac{\tau_1^3}{3!} + \cdots ,$$

$$\underset{\sim}{r_2} = \underset{\sim}{r_0} + \underset{\sim}{r_0}'\tau_2 + \underset{\sim}{r_0}''\frac{\tau_2^2}{2!} + \underset{\sim}{r_0}^{(3)}\frac{\tau_2^3}{3!} + \cdots .$$

We eliminate $\underset{\sim}{r_0}''$ using the equation (2.2). Then

$$\underset{\sim}{r_1} = Q_1\underset{\sim}{r_0} + \underset{\sim}{r_0}'\tau_1 + \underset{\sim}{r_0}^{(3)}\frac{\tau_1^3}{3!} + \cdots ,$$

$$\underset{\sim}{r_2} = Q_2\underset{\sim}{r_0} + \underset{\sim}{r_0}'\tau_2 + \underset{\sim}{r_0}^{(3)}\frac{\tau_2^3}{3!} + \cdots , \tag{2.7}$$

where $Q_1 = 1 - \frac{(1+m)\tau_1^2}{2r_0^3}$ and $Q_2 = 1 - \frac{(1+m)\tau_2^2}{2r_0^3}$.

Neglecting higher terms in the expansions (2.7), we obtain

$$\underset{\sim}{r_0}' = \left[(\underset{\sim}{r_1} - \underset{\sim}{r_0}Q_1)\tau_2^3 - (\underset{\sim}{r_2} - Q_2\underset{\sim}{r_0})\tau_1^3 \right] \Big/ \tau_1\tau_2(\tau_2^2 - \tau_1^2) . \tag{2.8}$$

Now, using the velocity vector $\underset{\sim}{r_0}'$ and the known position vector $\underset{\sim}{r_0}$ we evaluate a first approximation of μ_0, σ_0 and ε_0 . Then, with the help of the table I, we calculate approximate values of the f and g

series at the point $\tau = \tau_1$, say f_1 and g_1. Since we have these values, we solve the equation (2.4) to obtain a better approximation of the velocity vector $\underset{\sim}{r}_0'$

$$\underset{\sim}{r}_0' = \frac{1}{g_1}\underset{\sim}{r}_1 - \frac{f_1}{g_1}\underset{\sim}{r}_0 \quad .$$

Repeating the same process, we can obtain the desirable accuracy for $\underset{\sim}{r}_0'$ which, of course, is confined because we truncated the f and g series.

We return now to the Laplacian method.

The geometrical conditions of the problem are expressed by the equation

$$\underset{\sim}{r} = \rho\hat{\underset{\sim}{\rho}} - \underset{\sim}{R} \,, \tag{2.9}$$

where $\underset{\sim}{r} \equiv (x, y, z)$ is the heliocentric radius vector of the planet, $\underset{\sim}{R} \equiv (X, Y, Z)$ and $\underset{\sim}{\rho} \equiv (\xi, \eta, \zeta)$ are the topocentric position vectors of the Sun and the planet respectively with reference to the observer. The co-ordinates $x, y, z, X, Y, Z, \xi, \eta, \zeta$ are the equatorial rectangular co-ordinates of the bodies.

We differentiate the equation (2.9) twice with respect to τ

$$\underset{\sim}{r}' = \rho'\hat{\underset{\sim}{\rho}} + \rho\hat{\underset{\sim}{\rho}}' - \underset{\sim}{R}' \,, \tag{2.10}$$

$$\underset{\sim}{r}'' = \rho''\hat{\underset{\sim}{\rho}} + 2\rho'\hat{\underset{\sim}{\rho}}' + \rho\hat{\underset{\sim}{\rho}}'' - \underset{\sim}{R}'' \,. \tag{2.11}$$

We now impose the dynamical condition that the Earth and the minor planet move around the Sun in accordance with the law of gravitation. Neglecting the attraction of other planets, we have

$$\underset{\sim}{R}'' = -\frac{1+m_e}{R^3}\underset{\sim}{R} \,, \qquad \underset{\sim}{r}'' = -\frac{1+m}{r^3}\underset{\sim}{r} \,,$$

where m_e and m are the masses of the Earth and the minor planet. The quantities are expressed in astronomical units, solar masses and $1/k$ mean solar days.

Neglecting the mass of minor planet, as very small compared with the mass of the Sun, the second equation becomes

$$\underset{\sim}{r}'' = -\frac{1}{r^3}(\rho\hat{\underset{\sim}{\rho}} - \underset{\sim}{R}) .$$

We substitute the expressions for $\underset{\sim}{R}''$ and $\underset{\sim}{r}''$ into (2.11). Then

$$-\frac{\rho}{r^3}\hat{\underset{\sim}{\rho}} + \left(\frac{1}{r^3} - \frac{1+m_e}{R^3}\right)\underset{\sim}{R} = \rho''\hat{\underset{\sim}{\rho}} + 2\rho'\hat{\underset{\sim}{\rho}}' + \rho\hat{\underset{\sim}{\rho}}'' .$$

Multiplying both sides by $\cdot(\hat{\underset{\sim}{\rho}} \times \hat{\underset{\sim}{\rho}}')$ and $\cdot(\hat{\underset{\sim}{\rho}} \times \hat{\underset{\sim}{\rho}}'')$ we obtain

$$\left(\frac{1+m_e}{R^3} - \frac{1}{r^3}\right)D_1 = \rho D , \qquad (2.12)$$

$$\left(\frac{1+m_e}{R^3} - \frac{1}{r^3}\right)D_2 = 2\rho' D , \qquad (2.13)$$

where $D = [\hat{\underset{\sim}{\rho}}, \hat{\underset{\sim}{\rho}}', \hat{\underset{\sim}{\rho}}'']$, $D_1 = [\hat{\underset{\sim}{\rho}}, \underset{\sim}{R}, \hat{\underset{\sim}{\rho}}']$, $D_2 = [\hat{\underset{\sim}{\rho}}, \hat{\underset{\sim}{\rho}}'', \underset{\sim}{R}]$.

It is necessary to have one more equation involving r and ρ . Such an equation may be obtained by squaring the equation (2.9). Then

$$r^2 = \rho^2 + R^2 - 2(\hat{\underset{\sim}{\rho}} \cdot \underset{\sim}{R})\rho . \qquad (2.14)$$

We have now to evaluate the determinants D, D_1 and D_2. Suppose that the topocentric equatorial co-ordinates, α_i, δ_i $(i=1,2,3)$ have been obtained from three observations made at the instants t_1, t_2 and t_3. The direction cosines of the line from the Earth to the Sun at $t = t_i$ $(i=1,2,3)$ are given by

$$\lambda_i = \cos\delta_i \cos\alpha_i \;, \quad \mu_i = \cos\delta_i \sin\alpha_i \;, \quad \nu_i = \sin\delta_i \quad (i=1,2,3) . \tag{2.15}$$

The quantities λ_i, μ_i, ν_i are the components of the unit vector $\hat{\underset{\sim}{\rho}}_i$. Thus the vector $\hat{\underset{\sim}{\rho}}_i$ is known.

The position vectors of the Sun $\underset{\sim}{R}_i$ $(i=1,2,3)$ are obtained by interpolation from the equatorial rectangular co-ordinates of the Sun, tabulated in the Astronomical Ephemeris.

The next step in the evaluation of the determinants consists of the determination of the first and second derivatives of the vector $\hat{\underset{\sim}{\rho}}$ at an instant t_0 such that $t_1 < t_0 < t_3$. The vector $\hat{\underset{\sim}{\rho}}$ can be expanded as power series in τ, that is

$$\hat{\underset{\sim}{\rho}} = \hat{\underset{\sim}{\rho}}_0 + \hat{\underset{\sim}{\rho}}_0'\tau + \hat{\underset{\sim}{\rho}}_0''\frac{\tau^2}{2!} + \cdots \quad . \tag{2.16}$$

In the case of the minor planets the series converges rapidly for small values of τ.

The series (2.16) can be written in terms of λ, μ and ν as

$$\begin{aligned} \lambda &= \lambda_0 + \lambda_0'\tau + \lambda_0''\frac{\tau^2}{2!} + \cdots\cdots \;, \\ \mu &= \mu_0 + \mu_0'\tau + \mu_0''\frac{\tau^2}{2!} + \cdots\cdots \;, \\ \nu &= \nu_0 + \nu_0'\tau + \nu_0''\frac{\tau^2}{2!} + \cdots\cdots \;. \end{aligned} \tag{2.17}$$

If the observations are equally (or nearly equally) spaced, one takes, in practice, the time of the second observation as the epoch t_0. Then $\tau_2 = 0$ and $\lambda_0 = \lambda_2$, $\mu_0 = \mu_2$, $\nu_0 = \nu_2$.

If only three observations are given, the series (2.17) have to be truncated. To obtain the components of the vectors $\hat{\underset{\sim}{\rho}}'$ and $\hat{\underset{\sim}{\rho}}''$ we obtain the following systems of equations

$$\left.\begin{aligned} \lambda_1 &= \lambda_0 + \lambda_0' \tau_1 + \lambda_0'' \frac{\tau_1^2}{2!} \\ \lambda_3 &= \lambda_0 + \lambda_0' \tau_3 + \lambda_0'' \frac{\tau_3^2}{2!} \end{aligned}\right\}, \qquad (2.18)$$

$$\left.\begin{aligned} \mu_1 &= \mu_0 + \mu_0' \tau_1 + \mu_0'' \frac{\tau_1^2}{2!} \\ \mu_3 &= \mu_0 + \mu_0' \tau_3 + \mu_0'' \frac{\tau_3^2}{2!} \end{aligned}\right\}, \qquad (2.19)$$

$$\left.\begin{aligned} \nu_1 &= \nu_0 + \nu_0' \tau_1 + \nu_0'' \frac{\tau_1^2}{2!} \\ \nu_3 &= \nu_0 + \nu_0' \tau_3 + \nu_0'' \frac{\tau_3^2}{2!} \end{aligned}\right\}, \qquad (2.20)$$

Since $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_3, \mu_3, \nu_3)$, $(\lambda_0 = \lambda_2, \mu_0 = \mu_2, \nu_0 = \nu_2)$, τ_1 and τ_3 are given we derive $(\lambda_0', \mu_0', \nu_0')$ and $(\lambda_0'', \mu_0'', \nu_0'')$ by solving the equations (2.18), (2.19) and (2.20).

The values so obtained are approximate due to the truncation of the series (2.17). If more than three observations are available we can obtain more accurate values, writing more equations containing more terms. Of course, the number of equations, which we can use, is limited by the fact that the instants of the observations must be included in the interval of convergence of the series (2.16), and we can no longer neglect the action of the other planets on the motion of the Earth and the minor planet, since more observations require a longer interval of time. This must be small, especially, when the perturbations from Jupiter on the minor planet are large.

The choice of the time of the second observation as the initial epoch has been made with the assumption that the second observation is halfway (or nearly halfway) between those of the other two. The question now arises, what should be taken as the initial epoch, when this assumption is not valid, and why we make such an assumption.

Let us consider the power series expansions (2.17) about some epoch point $t_0 \neq t_2$. Then the errors, resulting from neglecting the higher terms in the right members, are given by the relations

$$\Delta\lambda_0 = -\frac{1}{3!}\lambda_0'''\tau_1\tau_2\tau_3 - \frac{1}{4!}\lambda_0^{(4)}\tau_1\tau_2\tau_3(\tau_1+\tau_2+\tau_3) + \cdots,$$

$$\Delta\lambda_0' = \frac{1}{3!}\lambda_0'''(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1) + \frac{1}{4!}\lambda_0^{(4)}(\tau_1+\tau_2)(\tau_2+\tau_3)(\tau_3+\tau_1) + \cdots, \quad (2.21)$$

$$\Delta\lambda_0'' = -\frac{1}{3!}\lambda_0'''(\tau_1+\tau_2+\tau_3) - \frac{1}{4!}\lambda_0^{(4)}(\tau_1^2+\tau_2^2+\tau_3^2+\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1) + \cdots,$$

and similar expressions for $\Delta\mu_0$, $\Delta\mu_0'$, $\Delta\mu_0''$, $\Delta\nu_0$, $\Delta\nu_0'$, $\Delta\nu_0''$.

Clearly, λ_0, μ_0, ν_0 are subjected to errors of the third order, λ_0', μ_0', ν_0' are subjected to errors of the second order, while the errors in λ_0'' , μ_0'' , ν_0'' are of the first order. Since, in general, an error of the first order is more serious than one of the second order, the origin of the time should be chosen as to make the first order errors in λ_0'' , μ_0'' , ν_0'' , equal to zero. This will be valid if

$$\tau_1+\tau_2+\tau_3=0 \;,$$

or

$$t_0=\frac{1}{3}(t_1+t_2+t_3). \tag{2.22}$$

Hence the instant of the second observation is taken to be the origin of the time when the successive observations are equally or about equally distant from one another.

The determinants D , D_1 and D_2 are easily computed at the point $\tau = 0$ from the evaluated derivatives $\hat{\underset{\sim}{\rho}}_0'$, $\hat{\underset{\sim}{\rho}}_0''$ and the known vectors $\underset{\sim}{R}_0$ and $\hat{\underset{\sim}{\rho}}_0$. Then, the equations (2.12) and (2.14) take the form

$$\rho_0=C+\frac{F}{r_0^3} \;, \tag{2.23}$$

$$r_0^2=\rho_0^2+A\cdot\rho_0+B \;, \tag{2.24}$$

where $C\equiv\left(\frac{1+m_e}{R_0^3}\right)\left(\frac{D_1}{D}\right)_{\tau=0}$, $F\equiv\left(\frac{-D_1}{D}\right)_{\tau=0}$, $A\equiv-2(\underset{\sim}{R}_0\cdot\hat{\underset{\sim}{\rho}}_0)$ and $B\equiv R_0^2$.

The index zero refers to the initial epoch t_0.

In order to find the values of r_0 and ρ_0 we have to solve the system of the equations (2.23), (2.24). Rather than attempting to solve this system by algebraic calculations, calculations which lead us to an equation of the eighth degree in ρ_0 , it is easier to obtain values for ρ_0 and r_0 by simultaneous solution of the equations of the system. This is accomplished by means of successive approximations. A value of r_0 about $2{,}8$ a.u., which is the mean heliocentric distance of the minor planets, is inserted in the equation (2.23) to obtain a value of ρ_0. Then with the help of the equation (2.24) we evaluate a second approximation of r_0. This is substituted into (2.23) and the process is continued until successive approximations are identical to the required accuracy.

After r_0 and ρ_0 are known we can evaluate ρ_0' making use of the equation (2.13), which at the point $\tau = 0$ becomes

$$\rho_0' = \left(\frac{1+m_e}{R_0^3} - \frac{1}{r_0^3}\right)\left(\frac{D_2}{2D}\right)_{\tau=0} . \tag{2.25}$$

The planetary aberration should now be taken into account. This is done subtracting from the E.T. the light time for the distance between the planet and the observer, that is

$$\text{corrected } t_i \equiv t_i^c = t_i - A^*\rho_i \quad (i=1,2,3) , \tag{2.26}$$

where $A^* = 0.00577560$ m.s.d., is the time, in mean solar days, during which the light travels one a.u.

For the modified variable time τ we have,

$$\text{corrected } \tau_i \equiv \tau_i^c = k(t_i^c - t_0^c) \quad (i=1,2,3) ,$$

which by the introduction of the equation (2.26) becomes

$$\tau_i^c = k(t_i - A^*\rho_i - t_0 + A^*\rho_0) \quad \text{or} \quad \tau_i^c = \tau_i - kA^*(\rho_i - \rho_0) \quad (i=1,2,3) . \tag{2.27}$$

In order to make these corrections to the instants of observations, it is necessary to know the values of ρ_i $(i = 1,2,3)$. These are given with sufficient approximation by

$$\rho_i = \rho_0 + \rho_0' \tau_i^c \quad (i=1,2,3) \; .$$

Substitution of this equation in equation (2.27) gives

$$\tau_i^c = \tau_i - K A^*(\rho_0 + \rho_0' \tau_i^c - \rho_0) \text{ or } \tau_i^c = \frac{\tau_i}{1 + A^* K \rho_0'}, \; (i=1,2,3). \quad (2.28)$$

With these new corrected values of $\tau_i^c \; (i=1,2,3)$ we repeat the evaluation of ρ_0 and ρ_0' until successive approximations of these are identical to the required accuracy.

We can now find the velocity vector $\underset{\sim}{r}_0'$. In order to do this we need the velocity vector $\underset{\sim}{R}_0'$. This is obtained by using the process which is described at the beginning of this chapter. From the known position vectors $\underset{\sim}{R}_1$, $\underset{\sim}{R}_0$, $\underset{\sim}{R}_3$ and the corresponding times $\tau_1, \tau_2 \equiv 0, \tau_3$ we calculate the values of the f and g series, and then making use of the equation (2.4) we obtain the velocity vector $\underset{\sim}{R}_0'$. Since ρ_0' , $\underset{\sim}{\hat{\rho}}_0'$ and $\underset{\sim}{R}_0'$ are already known, the equation (2.10), at the point $t = t_0$ gives the desired velocity vector $\underset{\sim}{r}_0'$.

The vectors $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_0'$ are already known, and as a matter of fact, their components comprise an alternative set of the six constants of integration of the two body problem, so we can proceed in the determination of the orbital elements of the minor planet.

Since the elements $\underset{\sim}{\Omega}$, i and ω are referred to the ecliptic, we rotate the co-ordinate system about x-axis through an angle ε , equal to the obliquity of the ecliptic. Then the components of the vectors $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_0'$ in the new system are given by

$$\begin{aligned} [x_e, y_e, z_e] &= [x, y, z]\, M(\varepsilon) \; , \\ [x_e', y_e', z_e'] &= [x', y', z']\, M(\varepsilon) \; , \end{aligned} \qquad (2.29)$$

where $\mathcal{M}(\varepsilon)$ is the matrix

$$\mathcal{M}(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} .$$

The angular momentum per unit mass

$$\underset{\sim}{h} = \underset{\sim}{r} \times \frac{d\underset{\sim}{r}}{dt} = k\left(\underset{\sim}{r} \times \frac{d\underset{\sim}{r}}{d\tau}\right) = k(\underset{\sim}{r} \times \underset{\sim}{r}') , \tag{2.30}$$

is constant and can be found at once. Its components in the ecliptic co-ordinate system are evaluated by

$$h_{x_e} = (y_e z_e' - y_e' z_e) , \quad h_{y_e} = (z_e x_e' - z_e' x_e) , \quad h_{z_e} = (x_e y_e' - x_e' y_e) . \tag{2.31}$$

On the other hand, the direction cosines of the unit vector $\hat{\underset{\sim}{h}}$ as is evident from the geometry of the orbit, are

$$\sin\Omega \sin i \; , -\cos\Omega \sin i \; , \; \cos i \; .$$

Since $\hat{\underset{\sim}{h}} = (h_{x_e}/h \; , \; h_{y_e}/h \; , \; h_{z_e}/h)$ it follows that

$$\frac{h_{x_e}}{h} = \sin\Omega \sin i \; , \; -\frac{h_{y_e}}{h} = \cos\Omega \sin i \; , \; \frac{h_{z_e}}{h} = \cos i \; . \tag{2.31}$$

From the formulae (2.31) , since h_{x_e} , h_{y_e} , h_{z_e} and h are known, we obtain $\underset{\sim}{\Omega}$ and i .

The semi-latus rectum of the orbit is found as a result of the relation

$$p = \frac{h^2}{G(m+m_\odot)} ,$$

where $h^2 = (h^2_{x_e} + h^2_{y_e} + h^2_{z_e})$, $m_\odot$ is the mass of the Sun and m the mass of the asteroid. Since m is much less than $m_\odot$ and, in general, unknown, it is usually neglected.

In order to find the eccentricity e let us consider the equation

$$r = p/(1+e\cos v) \quad \text{or} \quad e\cos v = \frac{p}{r} - 1 , \tag{2.32}$$

which describes, in first approximation, the relative motion of the asteroid. Differentiating this equation once with respect to t we obtain

$$-e\sin v\,\dot{v} = -\frac{p}{r^2}\dot{r}. \tag{2.33}$$

Since the tangential component of the velocity in a motion is $V = r\cdot\dot{v}$ and $h = r\cdot V$ the equation (2.33) becomes

$$e\sin v = \frac{p}{h}\dot{r}. \tag{2.34}$$

Squaring and adding the equations (2.32) and (2.34) we obtain

$$e^2 = \left(\frac{p}{h}\dot{r}\right)^2 + \left(\frac{p}{r} - 1\right)^2 , \tag{2.35}$$

which gives the value of the eccentricity. Using this value, with the help of the equations (2.32) and (2.34), we can find the true anomaly.

We notice that $\dot{r}$ is given by

$$\dot{r} = \frac{d\,(x^2+y^2+z^2)^{1/2}}{dt} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2(x\dot{x}+y\dot{y}+z\dot{z}),$$

or

$$\dot{r} = \frac{\underline{r}\cdot\dot{\underline{r}}}{r}\ .$$

The semi-major axis a is obtained directly from

$$a = \frac{p}{(1-e^2)}\ .$$

Next step is to find ω . The heliocentric ecliptic rectangular co-ordinates are given by

$$x_e = r\left(\cos\Omega\cos(\omega+v) - \sin\Omega\sin(\omega+v)\cos i\right),$$

$$y_e = r\left(\sin\Omega\cos(\omega+v) + \cos\Omega\sin(\omega+v)\cos i\right),$$

$$z_e = r\left(\sin(\omega+v)\sin i\right).$$

Multiplying the first equation by $\cos\Omega$, the second by $\sin\Omega$ and adding we obtain

$$x_e\cos\Omega + y_e\sin\Omega = r\cos(\omega+v)\ .$$

Similarly, multiplying the first equation by $-\sin\Omega\cos i$, the second by $\cos\Omega\cos i$, the third by $\sin i$ and adding we get

$$-x_e\sin\Omega\cos i + y_e\cos\Omega\cos i + z_e\sin i = r\sin(\omega+v)\ .$$

The combination of these two equations gives

$$\tan(\omega+v) = \frac{-x_e\sin\Omega\cos i + y_e\cos\Omega\cos i + z_e\sin i}{x_e\cos\Omega + y_e\sin\Omega}\ .$$

Since υ, Ω, i are already known, we obtain ω.

The mean daily motion n is obtained without difficulty from the relation

$$n^2 = k^2 / a^3 .$$

The already known true anomaly and the eccentric anomaly are connected by the relation

$$\tan\frac{E}{2} = \left(\frac{1-e}{1+e}\right)^{1/2} \tan\frac{\upsilon}{2} .$$

Consequently we can easily find the value of E.

Using Kepler's equation

$$M = E - e\sin E ,$$

we obtain the mean anomaly M and hence the time of perihelion passage, since

$$M = n(t-T) \quad \text{or} \quad T = t - \frac{M}{n} .$$

TABLE I. Coefficients f_n, g_n to 12th order.

$$f_0=1$$

$$f_1=0$$

$$f_2=-\mu$$

$$f_3=3\sigma\mu$$

$$f_4=-15\sigma^2\mu+3\epsilon\mu+\mu^2$$

$$f_5=105\sigma^3\mu-\sigma(45\epsilon\mu+15\mu^2)$$

$$f_6=-945\sigma^4\mu+\sigma^2(630\epsilon\mu+210\mu^2)-24\epsilon\mu^2-45\epsilon^2\mu-\mu^3$$

$$f_7=10395\sigma^5\mu-\sigma^3(9450\epsilon\mu+3150\mu^2)+\sigma(882\epsilon\mu^2+1575\epsilon^2\mu+63\mu^3)$$

$$\begin{aligned}f_8=&-135135\sigma^6\mu+\sigma^4(155925\epsilon\mu+51975\mu^2)\\&-\sigma^2(24570\epsilon\mu^2+42525\epsilon^2\mu+2205\mu^3)+117\epsilon\mu^3+1575\epsilon^3\mu\\&+1107\epsilon^2\mu^2+\mu^4\end{aligned}$$

$$\begin{aligned}f_9=&\,2027025\sigma^7\mu-\sigma^5(2837835\epsilon\mu+945945\mu^2)\\&+\sigma^3(644490\epsilon\mu^2+1091475\epsilon^2\mu+65835\mu^3)\\&-\sigma(10935\epsilon\mu^3+99225\epsilon^3\mu+74385\epsilon^2\mu^2+255\mu^4)\end{aligned}$$

$$\begin{aligned}f_{10}=&-34459425\sigma^8\mu+\sigma^6(56756700\epsilon\mu+18918900\mu^2)\\&-\sigma^4(17027010\epsilon\mu^2+28378350\epsilon^2\mu+1891890\mu^3)\\&+\sigma^2(599940\epsilon\mu^3+4365900\epsilon^3\mu+3421440\epsilon^2\mu^2+21120\mu^4)\\&-498\epsilon\mu^4-99225\epsilon^4\mu-85410\epsilon^3\mu^2-15066\epsilon^2\mu^3-\mu^5\end{aligned}$$

$$\begin{aligned}f_{11}=&\,654729075\sigma^9\mu-\sigma^7(1240539300\epsilon\mu+413513100\mu^2)\\&+\sigma^5(465404940\epsilon\mu^2+766215450\epsilon^2\mu+54864810\mu^3)\\&-\sigma^3(27027000\epsilon\mu^3+170270100\epsilon^3\mu+137837700\epsilon^2\mu^2\\&+1201200\mu^4)+\sigma(114444\epsilon\mu^4+9823275\epsilon^4\mu+9058500\epsilon^3\mu^2\\&+2023758\epsilon^2\mu^3+1023\mu^5)\end{aligned}$$

$$\begin{aligned}f_{12}=&-13749310575\sigma^{10}\mu+\sigma^8(29462808375\epsilon\mu+9820936125\mu^2)\\&-\sigma^6(13315121820\epsilon\mu^2+21709437750\epsilon^2\mu+1640268630\mu^3)\\&+\sigma^4(1122971850\epsilon\mu^3+6385128750\epsilon^3\mu+5298643350\epsilon^2\mu^2\\&+58108050\mu^4)-\sigma^2(12072060\epsilon\mu^4+638512875\epsilon^4\mu\\&+618918300\epsilon^3\mu^2+159729570\epsilon^2\mu^3+195195\mu^5)+2031\epsilon\mu^5\\&+9823275\epsilon^5\mu+9951525\epsilon^4\mu^2+2480958\epsilon^3\mu^3+164610\epsilon^2\mu^4+\mu^6\end{aligned}$$

$$g_0=0$$

$$g_1=1$$

$$g_2=0$$

$$g_3=-\mu$$

$$g_4=6\sigma\mu$$

$$g_5=-45\sigma^2\mu+9\epsilon\mu+\mu^2$$

$$g_6=420\sigma^3\mu-\sigma(180\epsilon\mu+30\mu^2)$$

$$g_7=-4725\sigma^4\mu+\sigma^2(3150\epsilon\mu+630\mu^2)-54\epsilon\mu^2-225\epsilon^2\mu-\mu^3$$

$$\begin{aligned}g_8=&\,62370\sigma^5\mu-\sigma^3(56700\epsilon\mu+12600\mu^2)\\&+\sigma(3024\epsilon\mu^2+9450\epsilon^2\mu+126\mu^3)\end{aligned}$$

$$\begin{aligned}g_9=&-945945\sigma^6\mu+\sigma^4(1091475\epsilon\mu+259875\mu^2)\\&-\sigma^2(111510\epsilon\mu^2+297675\epsilon^2\mu+6615\mu^3)+243\epsilon\mu^3+11025\epsilon^3\mu\\&+4131\epsilon^2\mu^2+\mu^4\end{aligned}$$

$$\begin{aligned}g_{10}=&\,16216200\sigma^7\mu-\sigma^5(22702680\epsilon\mu+5675670\mu^2)\\&+\sigma^3(3617460\epsilon\mu^2+8731800\epsilon^2\mu+263340\mu^3)\\&-\sigma(35100\epsilon\mu^3+793800\epsilon^3\mu+371790\epsilon^2\mu^2+510\mu^4)\end{aligned}$$

$$\begin{aligned}g_{11}=&-310134825\sigma^8\mu+\sigma^6(510810300\epsilon\mu+132432300\mu^2)\\&-\sigma^4(113513400\epsilon\mu^2+255405150\epsilon^2\mu+9459450\mu^3)\\&+\sigma^2(2589840\epsilon\mu^3+39293100\epsilon^3\mu+21116700\epsilon^2\mu^2+63360\mu^4)\\&-1008\epsilon\mu^4-893025\epsilon^4\mu-457200\epsilon^3\mu^2-50166\epsilon^2\mu^3-\mu^5\end{aligned}$$

$$\begin{aligned}g_{12}=&\,6547290750\sigma^9\mu-\sigma^7(12405393000\epsilon\mu+3308104800\mu^2)\\&+\sigma^5(3587023440\epsilon\mu^2+7662154500\epsilon^2\mu+329188860\mu^3)\\&-\sigma^3(145945800\epsilon\mu^3+1702701000\epsilon^3\mu+1005404400\epsilon^2\mu^2\\&+4804800\mu^4)+\sigma(355608\epsilon\mu^4+98232750\epsilon^4\mu+60350400\epsilon^3\mu^2\\&+9227196\epsilon^2\mu^3+2046\mu^5)\end{aligned}$$

Chapter 3
The Solution of the N-Body Problem by Power Series

The n-body problem is defined in the following way. Consider n bodies with masses $m_1, m_2, \cdots m_n$ with such an internal mass distribution that they may be considered as particles which attract each other according to the Newtonian law of gravitation. They are free to move in space, initially in any given manner. The problem is to find their subsequent motion.

The notation used in this chapter is given below.

The position vectors of the particles and their co-ordinates in a Cartesian right handed inertial system are:

$$\underset{\sim}{r_1} \equiv (x_1, y_1, z_1), \quad \underset{\sim}{r_2} \equiv (x_2, y_2, z_2), \quad \cdots \underset{\sim}{r_n} \equiv (x_n, y_n, z_n). \tag{3.1}$$

The vector connecting the i^{th} and j^{th} $(i, j = 1, \dots n, i \neq j)$ particles is:

$$\underset{\sim}{r_{ij}} = \underset{\sim}{r_i} - \underset{\sim}{r_j}. \tag{3.2}$$

The derivatives of the position vectors of the bodies, with respect to τ, are:

$$\underset{\sim}{r_i}^{(1)} \equiv \underset{\sim}{r_i}' \equiv \frac{d\underset{\sim}{r_i}}{d\tau}, \quad \underset{\sim}{r_i}^{(2)} \equiv \underset{\sim}{r_i}'' \equiv \frac{d^2\underset{\sim}{r_i}}{d^2\tau} \quad \text{and in general } \underset{\sim}{r_i}^{(\nu)} \equiv \frac{d^\nu \underset{\sim}{r_i}}{d^\nu \tau} \quad (i = 1, \cdots n), \ \nu \geqslant 1 \tag{3.3}$$

where $\tau = k(t - t_0)$. These derivatives, with respect to t, are:

$$\underset{\sim}{r_i}^{[1]} \equiv \dot{\underset{\sim}{r_i}} \equiv \frac{d\underset{\sim}{r_i}}{dt}, \quad \underset{\sim}{r_i}^{[2]} \equiv \ddot{\underset{\sim}{r_i}} \equiv \frac{d^2\underset{\sim}{r_i}}{d^2 t} \quad \text{and in general } \underset{\sim}{r_i}^{[\nu]} \equiv \frac{d^\nu \underset{\sim}{r_i}}{d^\nu t} \quad (i = 1, \cdots n). \ \nu \geqslant 1 \tag{3.4}$$

Hence the following relation is evident:

$$k^{\nu}\, \underset{\sim}{r}_i^{(\nu)} = \underset{\sim}{r}_i^{[\nu]} \qquad \nu \geqslant 1 \quad (i = 1, \cdots n)\,. \tag{3.5}$$

Neglecting any acceleration possibly caused by thrust, drag, gravitational forces exterior to the system of the bodies, etc., the force acting on the i^{th} body due to the other bodies is given by the relation

$$\underset{\sim}{F}_i = -k^2 m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{\underset{\sim}{r}_i - \underset{\sim}{r}_j}{r_{ij}^3} \qquad (i = 1, \cdots n)\,. \tag{3.6}$$

Since this force, according to Newton's second law, is equal to $m_i d^2 \underset{\sim}{r}_i / dt^2$ $(i = 1, \cdots n)$ equation (3.6) becomes

$$m_i \frac{d^2 \underset{\sim}{r}_i}{dt^2} = -k^2 m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{\underset{\sim}{r}_i - \underset{\sim}{r}_j}{r_{ij}^3} \qquad (i = 1, \cdots n)\,. \tag{3.7}$$

However, the form of equation (3.7) is not very useful for practical applications. We consider, therefore, an (n + 1)-body system, and for the sake of convenience the origin of the co-ordinate system is transferred to a particular body. The form of the equations of motion, referred to this relative co-ordinate system, fixed to one of the bodies, say to the $(n+1)^{th}$, are

$$\frac{d^2 \underset{\sim}{r}_i}{dt^2} = -k^2 (m_0 + m_i) \frac{\underset{\sim}{r}_i}{r_i^3} - k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \left(\frac{\underset{\sim}{r}_i - \underset{\sim}{r}_j}{r_{ij}^3} + \frac{\underset{\sim}{r}_j}{r_j^3} \right) \quad (i = 1, \cdots n)\,, \tag{3.8}$$

where k is the gravitational constant, $m_0 \equiv m_{n+1}$ is the mass of the central body which is considered as the origin of the co-ordinate system, m_i is the mass of the i^{th} body, r_i the distance of the i^{th} body from the origin and r_{ij} the distance between m_i and m_j. It is obvious that

$$r_i = |\underset{\sim}{r_i}| = (x_i^2 + y_i^2 + z_i^2)^{1/2} \qquad (i = 1, \dots n), \tag{3.9}$$

$$r_{ij} = |\underset{\sim}{r_j} - \underset{\sim}{r_i}| = \left[(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2\right]^{1/2} \quad (i,j = 1, \cdots n, \; i \neq j).$$

Since the system being considered is the Solar System, to establish the equations of motion using dimensionless variables, we express the distances in astronomical units, the masses in solar masses and the time in $1/k$ mean solar days. Then equation (3.8), by introduction of the notation (3.3) and the equation (3.5), becomes

$$\underset{\sim}{r_i}'' = -(1+m_i)\frac{\underset{\sim}{r_i}}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{\underset{\sim}{r_i} - \underset{\sim}{r_j}}{r_{ij}^3} + \frac{\underset{\sim}{r_j}}{r_j^3}\right) \qquad (i = 1, \cdots n), \tag{3.10}$$

or

$$\underset{\sim}{r_i}'' = \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right]\underset{\sim}{r_i} + \sum m_j \left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right)\underset{\sim}{r_j} \quad (i = 1, \cdots n).$$

The system (3.10) consists of n second-order differential equations for the vectors $\underset{\sim}{r_i}$ $(i = 1, 2 \cdots n)$ and it is equivalent to a system of 3n second-order differential equations for the co-ordinates x_i, y_i, z_i $(i = 1, \cdots n)$. That is

$$\begin{aligned}
x_i'' &= \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) x_j, \\
y_i'' &= \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] y_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) y_j, \quad (i = 1, \cdots n) \\
z_i'' &= \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] z_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) z_j.
\end{aligned} \tag{3.11}$$

We reduce this second-order system to another, of first order, by introducing new variables given by

$$\underset{\sim}{Q}_i=(u_i, v_i, w_i) \equiv (x_i', y_i', z_i') \qquad (i=1, \cdots n), \tag{3.12}$$

and we obtain

$$x_i' = u_i$$
$$y_i' = v_i$$
$$z_i' = w_i$$

$$u_i' = \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) x_j$$

$$v_i' = \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] y_i + \sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) y_j \qquad (i=1, \cdots n) \tag{3.13}$$

$$w_i' = \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \frac{1}{r_{ij}^3}\right] z_i + \sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left(\frac{1}{r_{ij}^3} - \frac{1}{r_j^3}\right) z_j \ .$$

In order to prove that there exists a solution of this system, three theorems are given below, which will be used for this purpose.

T h e o r e m 1 : If a function f has a derivative at a point τ_1 then it is also continuous at τ_1. (Apostol, Vol. I, p. 116)

T h e o r e m 2 : Let f and g be continuous functions at a point τ_1, then the sum $f+g$, the difference $f-g$ and the product $f \cdot g$ are also continuous at τ_1. The same is true for the quotient f/g if $g(\tau_1) \neq 0$. (Apostol, Vol. I, p.115).

T h e o r e m 3 : A general first-order system for m unknowns functions $\varphi_1, \varphi_2, \cdots \varphi_m$, may be written as:

$$\varphi_1' = f_1(\tau, \varphi_1, \varphi_2, \ldots \varphi_m)$$

$$\varphi_2' = f_2(\tau, \varphi_1, \varphi_2, \ldots \varphi_m) \tag{3.14}$$

$$\varphi_m' = f_m(\tau, \varphi_1, \varphi_2, \ldots \varphi_m).$$

Suppose that $f_1, f_2, \ldots f_m$ and the derivatives $\partial f_i / \partial \varphi_j$ $(i, j = 1, \ldots m)$ are continuous in a box

$$|\tau - \tau^\circ| < K,$$

$$|\varphi_i - \varphi_i^\circ| < K \quad (i = 1, \ldots m),$$

Where $\tau^\circ, \varphi_1^\circ, \ldots \varphi_m^\circ$ are given numbers. Then for some positive number h there exists a solution $\underset{\sim}{\varphi}(\tau) = (\varphi_1(\tau), \varphi_2(\tau), \ldots \varphi_m(\tau))$ of the system (3.14) which is defined for τ in the interval $|\tau - \tau^\circ| < h$ and satisfies the conditions $\varphi_i(\tau^\circ) = \varphi_i^\circ$ $(i = 1, \ldots m)$. The solution is unique in the sense that, if $\varphi_i^* = \varphi_i^*(\tau)$ $(i = 1, \ldots m)$ is another solution and satisfies $\varphi_i^*(\tau^\circ) = \varphi_i^\circ$ $(i = 1, \ldots m)$, then $\varphi_i^*(\tau) = \varphi_i(\tau)$ in their common interval of definition (Morrey, p. 698)

Firstly, we notice that the existence of the derivatives x_i', y_i', z_i', $u_i' \equiv x_i''$, $v_i' \equiv y_i''$, $w_i' \equiv z_i''$ $(i = 1, \ldots n)$ is obvious by the nature of the problem, and for the reason that the components of the velocity and acceleration vectors (to which the derivatives are identical) are limited for every value of the time τ. Similarly, the functions $r_i = r_i(\tau)$; $r_{ij} = r_{ij}(\tau)$ $(i, j = 1, \ldots n \; i \neq j)$, possess first order derivatives in the interval $(-\infty, +\infty)$, which are calculated as follows:

$$\underset{\sim}{r_i}\cdot\underset{\sim}{r_i} = r_i^2 \rightarrow 2\,\underset{\sim}{r_i}\cdot\underset{\sim}{r_i}' = 2\,r_i\,r_i' \rightarrow r_i' = \frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_i}'}{r_i} = \frac{x_i x_i' + y_i y_i' + z_i z_i'}{(x_i^2+y_i^2+z_i^2)^{1/2}}, \quad (3.15)$$

$$(i = 1, \ldots n),$$

$$\underset{\sim}{r_{ij}}\cdot\underset{\sim}{r_{ij}} = r_{ij}\,r_{ij} \rightarrow 2\,\underset{\sim}{r_{ij}}\cdot\underset{\sim}{r_{ij}}' = 2\,r_{ij}\,r_{ij}' \rightarrow r_{ij}' = \frac{\underset{\sim}{r_{ij}}'\,\underset{\sim}{r_{ij}}}{r_{ij}} = \frac{(\underset{\sim}{r_i}-\underset{\sim}{r_j})(\underset{\sim}{r_i}'-\underset{\sim}{r_j}')}{r_{ij}}, \quad (3.16)$$

hence

$$r_{ij}' = \frac{(x_j-x_i)(x_j'-x_i')+(y_j-y_i)(y_j'-y_i')+(z_j-z_i)(z_j'-z_i')}{\left((x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\right)^{1/2}} \qquad (i,j = 1,\ldots n \;\; i\neq j).$$

The functions $\quad r_i^{-3} = r_i^{-3}(\tau)\,,\; r_{ij}^{-3} = r_{ij}^{-3}(\tau) \qquad (i,j = 1,\ldots.n \quad i\neq j)$ therefore, possess first-order derivatives for every value of τ in the interval $(-\infty,+\infty)$. These are given by

$$(r_i^{-3})' = -3\,r_i^{-4}\,r_i'\,,\;\; (r_{ij}^{-3})' = -3\,r_{ij}^{-4}\,r_{ij}' \qquad (i,j=1,\ldots n \;\; i\neq j).$$

We must, obviously, exclude the points of singularity, if they exist.

Consequently, according to theorems 1 and 2, the right hand side terms of system (3.13) are continuous in the open interval $(-\infty,+\infty)$, except at the points of singularity.

In the same way we can easily prove that the partial derivatives of the right hand side terms of the system (3.13) are continuous in the same interval.

So the theorem 3 assures the existence of a unique solution of system (3.13).

The next step is to prove that the position vectors $\underset{\sim}{r_i}$ $(i = 1,\ldots n)$, being considered as functions of the modified time τ, possess derivatives of all orders and so the unique solution of the system (3.13) can be expanded in power series. For this purpose the following expressions are introduced

$$\alpha_i \equiv r_i^{-3} \quad , \quad \beta_{ijk} \equiv \frac{\underset{\sim}{r}_i' \cdot \underset{\sim}{r}_j}{r_k^2} \quad (i,j,k=1\cdots n) \ ,$$

$$\gamma_{ijk} \equiv \frac{\underset{\sim}{r}_i' \cdot \underset{\sim}{r}_j'}{r_k^2} \quad , \quad \delta_{ijk} \equiv \frac{\underset{\sim}{r}_i \cdot \underset{\sim}{r}_j}{r_k^2} \quad (i,j,k=1\cdots n) \ , \tag{3.17}$$

$$a_{ij} \equiv r_{ij}^{-3} \quad , \quad b_{ij} \equiv \frac{r_i^2}{r_{ij}^2} \quad (i,j=1\cdots n \ \ i\neq j) \ ,$$

where it is obvious that $\gamma_{ijk}=\gamma_{jik}$, $\delta_{ijk}=\delta_{jik}$ $(i,j,k=1\cdots n)$, $a_{ij}=a_{ji}$ $(i,j=1,\cdots n \ i\neq j)$ and $\delta_{iii} = 1$ $(i=1,\cdots n)$.

The proof that the derivatives of these expressions are functions of the expressions themselves is given below.

They are initially given the formulae for the acceleration vectors $\underset{\sim}{r}_i''$, $\underset{\sim}{r}_j''$ of the i^{th} and j^{th} particles by changing the indices of equation (3.10), a fact which will help us later to avoid confusion with the indices of the expressions (3.17)

$$\underset{\sim}{r}_i'' = -(1+m_i)\frac{\underset{\sim}{r}_i}{r_i^3} - \underset{\sim}{r}_i \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \frac{1}{r_{i\lambda}^3} + \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \frac{\underset{\sim}{r}_\lambda}{r_{i\lambda}^3} - \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \frac{\underset{\sim}{r}_\lambda}{r_\lambda^3} \quad (i=1\cdots n) \ , \tag{3.18}$$

$$\underset{\sim}{r}_j'' = -(1+m_j)\frac{\underset{\sim}{r}_j}{r_j^3} - \underset{\sim}{r}_j \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \frac{1}{r_{j\mu}^3} + \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \frac{\underset{\sim}{r}_\mu}{r_{j\mu}^3} - \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \frac{\underset{\sim}{r}_\mu}{r_\mu^3} \quad (j=1\cdots n) \ . \tag{3.19}$$

We proceed now to the calculation of the derivatives

$$\alpha_i' \ , \ \beta_{ijk}' \ , \ \gamma_{ijk}' \ , \ \delta_{ijk}' \quad (i,j,k=1\cdots n) \ , \ a_{ij}' \ , \ b_{ij}' \quad (i,j=1\ldots n \ \ i\neq j) \ .$$

Differentiating the first of the relations (3.17) once with respect to τ we obtain

$$\alpha_i' = (r_i^{-3})' = -3\, r_i' r_i^{-4} \quad (i = 1 \cdots n) ,$$

which by introduction of equation (3.15) becomes

$$\alpha_i' = -3\, r_i^{-4} \frac{\underset{\sim}{r_i} \cdot \underset{\sim}{r_i}'}{r_i} = -3\, r_i^{-3} \frac{\underset{\sim}{r_i} \cdot \underset{\sim}{r_i}'}{r_i^2} = -3\, \alpha_i \beta_{iii} \quad (i = 1 \cdots n) . \tag{3.20}$$

Similarly, differentiation of the second of the relations (3.17) gives

$$\beta_{ijk}' = \left(\frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}}{r_k^2} \right)' = \frac{\underset{\sim}{r_i}'' \cdot \underset{\sim}{r_j}}{r_k^2} + \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}'}{r_k^2} - 2\, \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}}{r_k^3}\, r_k' \quad (i, j, k = 1 \cdots n) .$$

Making use of equations (3.15) and (3.18) we obtain

$$\begin{aligned}
\beta_{ijk}' &= \frac{\underset{\sim}{r_j}}{r_k^2} \cdot \left[-(1+m_i) \frac{\underset{\sim}{r_i}}{r_i^3} - \underset{\sim}{r_i} \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \frac{1}{r_{i\lambda}^3} + \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \frac{\underset{\sim}{r_\lambda}}{r_{i\lambda}^3} - \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \frac{\underset{\sim}{r_\lambda}}{r_\lambda^3} \right] + \\
&\quad + \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}'}{r_k^2} - 2\, \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}}{r_k^3}\, \frac{\underset{\sim}{r_k}' \cdot \underset{\sim}{r_k}}{r_k} = \\
&= -(1+m_i)\, r_i^{-3} \frac{\underset{\sim}{r_i} \cdot \underset{\sim}{r_j}}{r_k^2} - \frac{\underset{\sim}{r_i} \cdot \underset{\sim}{r_j}}{r_k^2} \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda r_{i\lambda}^{-3} + \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda r_{i\lambda}^{-3} \frac{\underset{\sim}{r_j} \cdot \underset{\sim}{r_\lambda}}{r_k^2} - \\
&\quad - \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda r_\lambda^{-3} \frac{\underset{\sim}{r_j} \cdot \underset{\sim}{r_\lambda}}{r_k^2} + \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}'}{r_k^2} - 2\, \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}}{r_k^2}\, \frac{\underset{\sim}{r_k}' \cdot \underset{\sim}{r_k}}{r_k^2} \quad (i, j, k = 1 \cdots n) ,
\end{aligned}$$

and according to the expressions (3.17)

$$\begin{aligned}
\beta_{ijk}' &= -(1+m_i)\, \alpha_i \delta_{ijk} - \delta_{ijk} \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \alpha_{i\lambda} + \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \alpha_{i\lambda} \delta_{j\lambda k} - \\
&\quad - \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \alpha_\lambda \delta_{j\lambda k} + \gamma_{ijk} - 2\, \beta_{ijk} \beta_{kkk} \quad (i, j, k = 1 \cdots n) .
\end{aligned} \tag{3.21}$$

Differentiating both sides of the third of the relations (3.17) with respect τ we obtain

$$\gamma_{ijk}' = \left(\frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}'}{r_k^2} \right)' = \frac{\underset{\sim}{r_i}'' \cdot \underset{\sim}{r_j}'}{r_k^2} + \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}''}{r_k^2} - 2\, \frac{\underset{\sim}{r_i}' \cdot \underset{\sim}{r_j}'}{r_k^3}\, r_k' ,$$

which upon substitution of equations (3.15), (3.18) and (3.19) becomes

$$\gamma'_{ijk} = \frac{\underset{\sim}{r}_j'}{r_k^2}\cdot\left[-(1+m_i)\frac{\underset{\sim}{r}_i}{r_i^3} - \underset{\sim}{r}_i \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \frac{1}{r_{i\lambda}^3} + \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \frac{\underset{\sim}{r}_\lambda}{r_{i\lambda}^3} - \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \frac{\underset{\sim}{r}_\lambda}{r_\lambda^3}\right] +$$
$$+ \frac{\underset{\sim}{r}_i'}{r_k^2}\cdot\left[-(1+m_j)\frac{\underset{\sim}{r}_j}{r_j^3} - \underset{\sim}{r}_j \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \frac{1}{r_{j\mu}^3} + \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \frac{\underset{\sim}{r}_\mu}{r_{j\mu}^3} - \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \frac{\underset{\sim}{r}_\mu}{r_\mu^3}\right] -$$
$$-2\,\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_j'}{r_k^3}\,\frac{\underset{\sim}{r}_k\cdot\underset{\sim}{r}_k'}{r_k} =$$

$$= -(1+m_i)\, r_i^{-3}\frac{\underset{\sim}{r}_j'\cdot\underset{\sim}{r}_i}{r_k^2} - \frac{\underset{\sim}{r}_j'\cdot\underset{\sim}{r}_i}{r_k^2}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda r_{i\lambda}^{-3} + \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda r_{i\lambda}^{-3}\frac{\underset{\sim}{r}_j'\cdot\underset{\sim}{r}_\lambda}{r_k^2} - \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda r_\lambda^{-3}\frac{\underset{\sim}{r}_j'\cdot\underset{\sim}{r}_\lambda}{r_k^2} -$$
$$-(1+m_j)\, r_j^{-3}\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_j}{r_k^2} - \frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_j}{r_k^2}\sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu r_{j\mu}^{-3} + \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu r_{j\mu}^{-3}\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_\mu}{r_k^2} - \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu r_\mu^{-3}\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_\mu}{r_k^2} -$$
$$-2\,\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_j'}{r_k^2}\,\frac{\underset{\sim}{r}_k\cdot\underset{\sim}{r}_k'}{r_k^2} \qquad (i,j,k=1,\ldots n).$$

Introduction of expressions (3.17) gives

$$\gamma'_{ijk} = -(1+m_i)\alpha_i \beta_{jik} - \beta_{jik}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha_{i\lambda} + \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha_{i\lambda}\beta_{j\lambda k} - \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha_\lambda \beta_{j\lambda k} -$$
$$-(1+m_j)\alpha_j \beta_{ijk} - \beta_{ijk}\sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \alpha_{j\mu} + \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \alpha_{j\mu}\beta_{i\mu k} - \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \alpha_\mu \beta_{i\mu k} - \qquad (3.22)$$
$$-2\gamma_{ijk}\beta_{kkk} \qquad (i,j,k=1\cdots n).$$

To find the derivative δ'_{ijk} $(i,j,k=1,2,\ldots n)$ we differentiate both sides of the relation

$$\delta_{ijk} = \frac{\underset{\sim}{r}_i\cdot\underset{\sim}{r}_j}{r_k^2} \qquad (i,j,k=1\cdots n)$$

with respect to τ to give

$$\delta'_{ijk} \equiv \left(\frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_j}}{r_k^2}\right)' = \frac{\underset{\sim}{r_i}'\cdot\underset{\sim}{r_j}}{r_k^2} + \frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_j}'}{r_k^2} - 2\,\frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_j}}{r_k^3}\, r_k' =$$

$$= \frac{\underset{\sim}{r_i}'\cdot\underset{\sim}{r_j}}{r_k^2} + \frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_j}'}{r_k^2} - 2\,\frac{\underset{\sim}{r_i}\cdot\underset{\sim}{r_j}}{r_k^2}\,\frac{\underset{\sim}{r_k}'\cdot\underset{\sim}{r_k}}{r_k^2} = \tag{3.23}$$

$$= b_{ijk} + b_{jik} - 2\,\delta_{ijk}\, b_{kkk} \quad (i, j, k = 1 \cdots n) .$$

Similarly, from the relation $\alpha_{ij} \equiv r_{ij}^{-3} \quad (i, j = 1, \cdots n \; i \neq j)$ we obtain

$$\alpha'_{ij} \equiv (r_{ij}^{-3})' = -3\, r_{ij}^{-4}\, r_{ij}' ,$$

which upon substitution of equation (3.16) becomes

$$\alpha'_{ij} = -3\, r_{ij}^{-4}\, \frac{(\underset{\sim}{r_j}-\underset{\sim}{r_i})\cdot(\underset{\sim}{r_j}'-\underset{\sim}{r_i}')}{r_{ij}} =$$

$$= -3\, r_{ij}^{-3}\, \frac{r_i^2}{r_{ij}^2}\; \frac{\underset{\sim}{r_j}'\cdot\underset{\sim}{r_j} - \underset{\sim}{r_i}'\cdot\underset{\sim}{r_j} - \underset{\sim}{r_j}'\cdot\underset{\sim}{r_i} + \underset{\sim}{r_i}'\underset{\sim}{r_i}}{r_i^2} =$$

$$= -3\, r_{ij}^{-3}\, \frac{r_i^2}{r_{ij}^2}\left[\frac{\underset{\sim}{r_j}'\cdot\underset{\sim}{r_j}}{r_i^2} - \frac{\underset{\sim}{r_i}'\cdot\underset{\sim}{r_j}}{r_i^2} - \frac{\underset{\sim}{r_j}'\cdot\underset{\sim}{r_i}}{r_i^2} + \frac{\underset{\sim}{r_i}'\cdot\underset{\sim}{r_i}}{r_i^2}\right] .$$

This, according to expressions (3.17), takes the form

$$\alpha'_{ij} = -3\,\alpha_{ij}\, b_{ij}\,(b_{jji} - b_{iji} - b_{jii} + b_{iii}) \quad (i, j = 1, \cdots n \; i \neq j) . \tag{3.24}$$

We obtain the derivative $b'_{ij} \quad (i, j = 1, \cdots n \; i \neq j)$ from the following

$$b'_{ij} = \left(\frac{r_i^2}{r_{ij}^2}\right)' = 2\,\frac{r_i\, r_i'}{r_{ij}^2} - 2\,\frac{r_i^2}{r_{ij}^3}\, r_{ij}' ,$$

which with the help of the equations (3.15) and (3.16) becomes

$$b_{ij}' = 2\,\frac{r_i \dfrac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_i}{r_i}}{r_{ij}^2} - 2\,\frac{r_i^2}{r_{ij}^3}\,\frac{(\underset{\sim}{r}_j-\underset{\sim}{r}_i)\cdot(\underset{\sim}{r}_j'-\underset{\sim}{r}_i')}{r_{ij}} =$$

$$= 2\,\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_i}{r_{ij}^2} - 2\,\frac{r_i^4}{r_{ij}^4}\,\frac{(\underset{\sim}{r}_j-\underset{\sim}{r}_i)\cdot(\underset{\sim}{r}_j'-\underset{\sim}{r}_i')}{r_i^2} =$$

$$= 2\,\frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_i}{r_i^2}\,\frac{r_i^2}{r_{ij}^2} - 2\left(\frac{r_i^2}{r_{ij}^2}\right)^2\left(\frac{\underset{\sim}{r}_j'\cdot\underset{\sim}{r}_j}{r_i^2} - \frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_j}{r_i^2} - \frac{\underset{\sim}{r}_i\cdot\underset{\sim}{r}_j'}{r_i^2} + \frac{\underset{\sim}{r}_i'\cdot\underset{\sim}{r}_i}{r_i^2}\right).$$

Introduction of expressions (3.17) gives

$$b_{ij}' = 2\,b_{ij}\beta_{iii} - 2\,b_{ij}^2\,(\beta_{jji} - \beta_{iji} - \beta_{jii} + \beta_{iii}) \quad (i,j=1\cdots n \;\; i\neq j). \tag{3.25}$$

For the sake of convenience, the results of the above calculations are summarized in the following

$$\alpha_i' = -3\,\alpha_i\,\beta_{iii} \quad (i=1,\cdots n),$$

$$\beta_{ijk}' = -(1+m_i)\,\alpha_i\,\delta_{ijk} - \delta_{ijk}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda} + \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda}\delta_{j\lambda k} -$$
$$-\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_\lambda\delta_{j\lambda k} + \gamma_{ijk} - 2\,\beta_{ijk}\beta_{kkk} \quad (i,j,k=1\cdots n),$$

$$\begin{aligned}\gamma_{ijk}' = &-(1+m_i)\alpha_i\beta_{jik} - \beta_{jik}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda} + \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda}\beta_{j\lambda k} - \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda\alpha_\lambda\beta_{j\lambda k} -\\ &-(1+m_j)\alpha_j\beta_{ijk} - \beta_{ijk}\sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu\alpha_{j\mu} + \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu\alpha_{j\mu}\beta_{i\mu k} - \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu\alpha_\mu\beta_{i\mu k} -\\ &-2\,\gamma_{ijk}\beta_{kkk} \quad (i,j,k=1\cdots n),\end{aligned} \tag{3.26}$$

$$\delta_{ijk}' = \beta_{ijk} + \beta_{jik} - 2\,\delta_{ijk}\beta_{kkk} \quad (i,j,k=1\cdots n),$$

$$\alpha_{ij}' = -3\,\alpha_{ij}\,b_{ij}\,(\beta_{jji} - \beta_{iji} - \beta_{jii} + \beta_{iii}) \quad (i,j=1\cdots n \;\; i\neq j),$$

$$b_{ij}' = -2\,b_{ij}\big(b_{ij}(\beta_{jji} - \beta_{iji} - \beta_{jii} + \beta_{iii}) - \beta_{iii}\big) \quad (i,j=1\cdots n \;\; i\neq j).$$

Differentiation of equations (3.26) with respect to τ, and the introduction of the same equations, as soon as the derivatives α_i', β_{ijk}', γ_{ijk}', δ_{ijk}' $(i,j,k=1\cdots n)$, α_{ij}', b_{ij}' $(i,j=1\cdots n\ i\neq j)$ appear, gives the second-order derivatives of the expressions (3.17) as functions of the expressions.

In order to find the higher-order derivatives of the expressions (3.17), we can repeat this differentiation process, each time eliminating α_i', β_{ijk}', γ_{ijk}', δ_{ijk}' $(i,j,k=1\cdots n)$, α_{ij}', b_{ij}' $(i,j=1\cdots n\ i\neq j)$ by means of equations (3.26) as the case demands.

As the next step in the course of the proof it will be shown that, by making use of expressions (3.17), we can actually obtain the derivatives $\underset{\sim}{r}_i^{(\nu)}$ $\quad \nu \geqslant 3 \quad (i=1,\cdots n)$ as functions of the position and the velocity vectors of the bodies $\underset{\sim}{r}_i$, $\underset{\sim}{r}_i'$ $(i=1,\cdots n)$.

The equations of motion already give the second-order derivatives $\underset{\sim}{r}_i''$ $(i=1,\cdots n)$ in the desired form. On substituting expressions (3.17) into equation (3.10) we obtain

$$\underset{\sim}{r}_i''=\left[-(1+m_i)\alpha_i-\sum_{\substack{j=1\\j\neq i}}^{n} m_j\alpha_{ij}\right]\underset{\sim}{r}_i+\sum_{\substack{j=1\\j\neq i}}^{n} m_j(\alpha_{ij}-\alpha_j)\underset{\sim}{r}_j \quad (i=1,..n). \tag{3.27}$$

Introducing a more general notation we can write this as follows

$$\underset{\sim}{r}_i''=\sum_{j=1}^{n}\left(C_{ij2}\underset{\sim}{r}_j+D_{ij2}\underset{\sim}{r}_j'\right) \quad (i=1,\cdots n), \tag{3.28}$$

where

$$\begin{aligned} C_{ii2} &\equiv -(1+m_i)\alpha_i-\sum_{\substack{j=1\\j\neq i}}^{n} m_j\alpha_{ij} \quad (i=1\cdots n), \\ C_{ij2} &\equiv m_j(\alpha_{ij}-\alpha_j) \quad (i,j=1,\cdots n\ \ i\neq j), \\ D_{ij2} &\equiv 0 \quad (i,j=1,\cdots n). \end{aligned} \tag{3.29}$$

The third-order derivative $\underset{\sim}{r}_i^{(3)}$ $(i=1,\cdots n)$ is easily calculated by differentiating equation (3.27)

$$\underset{\sim}{r}_i^{(3)}=\left[-(1+m_i)\alpha_i'-\sum_{\substack{j=1\\j\neq i}}^{n}m_j\alpha_{ij}'\right]\underset{\sim}{r}_i+\sum_{\substack{j=1\\j\neq i}}^{n}m_j(\alpha_{ij}'-\alpha_j')\underset{\sim}{r}_j+$$

$$+\left[-(1+m_i)\alpha_i-\sum_{\substack{j=1\\j\neq i}}^{n}m_j\alpha_{ij}\right]\underset{\sim}{r}_i'+\sum_{\substack{j=1\\j\neq i}}^{n}m_j(\alpha_{ij}-\alpha_j)\underset{\sim}{r}_j' \quad (i=1,\cdots n),$$

which upon substitution of equations (3.26) becomes

$$\begin{aligned}\underset{\sim}{r}_i^{(3)}=\Big[&3(1+m_i)\alpha_i b_{iii}+\\&+3\sum_{\substack{j=1\\j\neq i}}^{n}m_j\alpha_{ij}b_{ij}(b_{iii}-b_{jii}-b_{iji}+b_{jji})\Big]\underset{\sim}{r}_i-\\&-3\sum_{\substack{j=1\\j\neq i}}^{n}m_j\left(\alpha_{ij}b_{ij}(b_{iii}-b_{jii}-b_{iji}+b_{jji})-\alpha_j b_{jjj}\right)\underset{\sim}{r}_j+\\&+\Big(-(1+m_i)\alpha_i-\sum_{\substack{j=1\\j\neq i}}^{n}m_j\alpha_{ij}\Big)\underset{\sim}{r}_i'+\sum_{\substack{j=1\\j\neq i}}^{n}m_j(\alpha_{ij}-\alpha_j)\underset{\sim}{r}_j' \quad (i=1,\cdots n).\end{aligned} \tag{3.30}$$

We may write this in the general form of equation (3.28), that is

$$\underset{\sim}{r}_i^{(3)}=\sum_{j=1}^{n}\left(C_{ij3}\underset{\sim}{r}_j+D_{ij3}\underset{\sim}{r}_j'\right) \quad (i=1,\cdots n), \tag{3.31}$$

where

$$C_{ii3}\equiv 3\left[(1+m_i)\alpha_i b_{iii}+\sum_{\substack{j=1\\j\neq i}}^{n}m_j\alpha_{ij}b_{ij}(b_{iii}-b_{jii}-b_{iji}+b_{jji})\right] \quad (i=1\cdots n)$$

$$C_{ij3} \equiv -3m_j\left(\alpha_{ij}b_{ij}(b_{iii}-b_{jii}-b_{iji}+b_{jji})-\alpha_j b_{jjj}\right) \quad (i,j=1,\cdots n \;\; i\neq j)\,,$$

$$D_{ii3} \equiv -(1+m_i)\alpha_i-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\alpha_{ij} \quad (i=1,\cdots n)\,,$$

$$D_{ij3} \equiv m_j(\alpha_{ij}-\alpha_j) \quad (i,j=1,\cdots n \;\; i\neq j)\,.$$

Having differentiated equation (3.30) once with respect to τ and having made use of equations (3.26) and (3.27), as soon as the derivatives $\alpha_i',\ b_{ijk}',\ \gamma_{ijk}',\ \delta_{ijk}',\ \underset{\sim}{r}_i''\ (i,j,k=1,\cdots n),\ \alpha_{ij}',\ b_{ij}'\ (i,j=1,\cdots n\ i\neq j)$ appear (and after substantial algebraic calculations), the fourth-order derivative $\underset{\sim}{r}_i^{(4)}\ (i=1,\cdots n)$ is obtained

$$\begin{aligned}
\underset{\sim}{r}_i^{(4)} = \Bigg\{ & (1+m_i)\alpha_i\cdot(-15b_{iii}^2-2(1+m_i)\alpha_i+3\gamma_{iii})+ \\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\Bigg[(1+m_i)m_j\alpha_i\left(-\alpha_{ij}+3(\alpha_{ij}-\alpha_j)\delta_{iji}\right)+ \\
&+3m_j\alpha_{ij}b_{ij}\Big(-5b_{ij}(b_{iii}-b_{jii}-b_{iji}+b_{jji})^2+\gamma_{iii}-\gamma_{iji}-\gamma_{jii}+\gamma_{jji}+ \\
&+(1+m_i)\alpha_i(\delta_{iji}-1)+(1+m_j)\alpha_j(\delta_{jii}-\delta_{jji})+ \\
&+m_\lambda\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n}\left(\alpha_{i\lambda}(\delta_{i\lambda i}-\delta_{j\lambda i}+\delta_{iji}-1)+\alpha_\lambda(\delta_{j\lambda i}-\delta_{i\lambda i})\right)+ \\
&+m_\mu\sum_{\substack{\mu=1\\ \mu\neq j}}^{n}\left(\alpha_{j\mu}(\delta_{j\mu i}-\delta_{i\mu i}+\delta_{jii}-\delta_{jji})+\alpha_\mu(\delta_{i\mu i}-\delta_{j\mu i})\right)\Big)+ \\
&+m_im_j(\alpha_{ij}-\alpha_j)(\alpha_{ji}-\alpha_i)+m_j\alpha_{ij}\sum_{\substack{j=1\\ j\neq i}}^{n}m_j\alpha_{ij}\Bigg]\Bigg\}\underset{\sim}{r}_i- \\
&-\sum_{\substack{j=1\\ j\neq i}}^{n}\Bigg\{3\Big[\alpha_i(4b_{iii}^2+(1+m_i)\alpha_i-\gamma_{iii})+\alpha_j b_{jjj}^2+ \\
&+\alpha_i\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n}m_\lambda(\alpha_{i\lambda}-(\alpha_{i\lambda}-\alpha_\lambda)\delta_{i\lambda i})+
\end{aligned}$$

$$
\begin{aligned}
&+3m_j\alpha_{ij}b_{ij}\Big(-5b_{ij}(\beta_{iii}-\beta_{jii}-\beta_{iji}+\beta_{jji})^2+\gamma_{jjj}-\gamma_{iji}-\gamma_{jii}+\gamma_{jji}+\\
&+(1+m_i)\alpha_i\delta_{iji}+(1+m_j)\alpha_j(\delta_{jii}-\delta_{jji}-1)+2(\beta_{iii}^2-\beta_{jjj}^2)+ \qquad\qquad (3.32)\\
&+m_\lambda\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n}\big(\alpha_{i\lambda}(\delta_{iji}-\delta_{j\lambda i})+\alpha_\lambda\cdot\delta_{j\lambda i}\big)+\\
&+m_\mu\sum_{\substack{\mu=1\\ \mu\neq j}}^{n}\big(\alpha_{j\mu}(\delta_{j\mu i}-\delta_{i\mu i}+\delta_{j\mu j}+\delta_{jii}-\delta_{jji}-1)+\alpha_\mu(\delta_{i\mu i}-\delta_{j\mu i}-\delta_{j\mu j})\big)\Big)\Big]+\\
&+m_j(\alpha_{ij}-\alpha_j)\Big((1+m_i)\alpha_i+(1+m_j)\alpha_j+\\
&+\sum_{\substack{\lambda\neq i\\ \lambda=1}}^{n}m_\lambda\alpha_{i\lambda}+\sum_{\substack{\mu=1\\ \mu\neq j}}^{n}m_\mu\alpha_{j\mu}\Big)\Big\}\,\underset{\sim}{r_j}+\\
&+\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n}\Big(m_\lambda(\alpha_{i\lambda}-\alpha_\lambda)\sum_{\substack{j=1\\ j\neq i\\ j\neq\lambda}}^{n}m_\lambda(\alpha_{\lambda j}-\alpha_j)\,\underset{\sim}{r_j}\Big)+\\
&+6\Big[(1+m_i)\alpha_i\beta_{iii}+\sum_{\substack{j=1\\ j\neq i}}^{n}m_j\alpha_{ij}b_{ij}(\beta_{iii}-\beta_{jii}-\beta_{iji}+\beta_{jji})\Big]\,\underset{\sim}{r_i}'-\\
&-6\sum_{\substack{j=1\\ j\neq i}}^{n}\Big[m_j\big(\alpha_{ij}b_{ij}(\beta_{iii}-\beta_{jii}-\beta_{iji}+\beta_{jji})-\alpha_i\beta_{iii}\big)\Big]\,\underset{\sim}{r_j}'\,. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=1,\ldots,n)\,.
\end{aligned}
$$

Equation (3.32) obviously has the form

$$\underset{\sim}{r_i}^{(4)}=\sum_{j=1}^{n}\left(C_{ij4}\,\underset{\sim}{r_j}+D_{ij4}\,\underset{\sim}{r_j}'\right)\quad (i=1\cdots n).$$

In a completely analogous manner, and since the expressions (3.17) are functions of the position and the velocity vectors of the bodies, one can find the derivatives of all orders of the position vectors, $\underset{\sim}{r_i}$ $(i=1\cdots n)$, as functions of the position and velocity vectors of the bodies. It is also evident that these derivatives have the general form

$$\underset{\sim}{\Gamma}_i^{(\nu)} = \sum_{j=1}^{n} \left(C_{ij\nu} \underset{\sim}{\Gamma}_j + D_{ij\nu} \underset{\sim}{\Gamma}_j' \right) \quad (i=1\cdots n) \quad \nu \geqslant 2, \quad (3.33)$$

and are continuous as algebraic combinations of continuous functions (see theorem 2).

The indices i, j, ν mean that, each $C_{ij\nu}$ or $D_{ij\nu}$ is a different expression of $a_i, b_{ijk}, \gamma_{ijk}, \delta_{ijk}$ $(i, j, k = 1, \cdots n)$, α_{ij}, b_{ij} $(i,j=1\cdots n \; i \neq j)$ for different $\underset{\sim}{\Gamma}_i$ $(i = 1 \cdots n)$ or different $\underset{\sim}{\Gamma}_j$ $(j = 1, 2, \cdots n)$ or different order of derivative.

If we consider $\underset{\sim}{\Gamma}_i^{(0)}$ $(i = 1, 2, \cdots n)$ to mean $\underset{\sim}{\Gamma}_i$ $(i = 1, \cdots n)$ then we obtain

$$\underset{\sim}{\Gamma}_i^{(\nu)} = \sum_{j=1}^{n} \left(C_{ij\nu} \underset{\sim}{\Gamma}_j + D_{ij\nu} \underset{\sim}{\Gamma}_j' \right) \quad (i=1, \cdots n) \quad \nu \geqslant 0 . \quad (3.34)$$

Comparing equation (3.34), for $\nu = 0$, with the equation $\underset{\sim}{\Gamma}_i^{(0)} \equiv \underset{\sim}{\Gamma}_i$ $(i = 1, \cdots n)$ it follows that the "coefficients" C_{ij0}, D_{ij0} have the values

$$\left.\begin{aligned} C_{ij0} &= \delta_{ij} \quad (i, j = 1, \cdots n), \\ D_{ij0} &= 0 \quad (i, j = 1, \cdots n), \end{aligned}\right\} \quad \tau \in (-\infty, +\infty) \quad (3.35)$$

where δ_{ij} is the Kronecker's delta.

Similarly, comparing equation (3.34), for $\nu = 1$, with the equation $\underset{\sim}{r}_i^{(1)} \equiv \underset{\sim}{r}_i'$ $(i = 1, \cdots n)$ gives the values of C_{ij1}, D_{ij1}

$$C_{ij1} = 0 \qquad (i, j = 1, \cdots n), \qquad \tau \in (-\infty, +\infty)$$
$$D_{ij1} = \delta_{ij} \qquad (i, j = 1, \cdots n). \tag{3.36}$$

We have already proved that the solution $\underset{\sim}{r}_i(\tau)$ $(i=1, \cdots n)$ of the system (3.13) is defined in the open interval $(-\infty, +\infty)$, (except at the points of singularity), and all the derivatives of this exist in the same interval. Hence, we can form the Taylor's series for $\underset{\sim}{r}_i(\tau)$ $(i=1, \cdots n)$ in powers of the independent variable τ about some epoch point, say $\underset{\sim}{r}_{i0}$ $(i = 1, \cdots n)$.

$$\underset{\sim}{r}_i = \sum_{\nu=0}^{\infty} \underset{\sim}{r}_{i0}^{(\nu)} \frac{\tau^\nu}{\nu!} \qquad (i = 1, \cdots n). \tag{3.37}$$

The proof that this series converges in an open interval $(-\tau^*, \tau^*)$ is given in the appendix A-1.

In equation (3.37) we interpret $\underset{\sim}{r}_{i0}^{(0)}$ $(i = 1, \cdots n)$ to mean $\underset{\sim}{r}_i$ $(i = 1, \cdots n)$ at $\tau = 0$ and $\underset{\sim}{r}_{i0}^{(\nu)} \equiv \left(d^\nu \underset{\sim}{r}_i / d\tau^\nu\right)_{\tau=0} (i=1, \cdots n)$, $\nu \geqslant 1$.

Later it will be shown that recursion formulae can be derived, which will provide numerical values for the power series coefficients. These formulae will enable us to compute $\underset{\sim}{r}_{i0}^{(\nu)}/\nu!$ $(i = 1, \cdots n)$ $\nu \geqslant 2$ successively in terms of $\underset{\sim}{r}_{i0}$ and $\underset{\sim}{r}_{i0}'$ $(i = 1, \cdots n)$. So as a first step we have to find formulae for the calculation of the velocity vectors $\underset{\sim}{r}_{i0}'$ $(i = 1, \cdots n)$.

Equation (3.34) gives the general form of the derivatives $\underset{\sim}{r}_i^{(\nu)}$ $(i=1, \ldots n)$ $\nu \geqslant 0$ for every value of the time τ. In the case of $\tau = 0$ we obtain

$$\underset{\sim}{r}_{io}^{(\nu)} = \sum_{j=1}^{n} \left(C_{ij\nu o}\, \underset{\sim}{r}_{jo} + D_{ij\nu o}\, \underset{\sim}{r}_{jo}' \right) \quad (i = 1, \cdots n) \quad \nu \geqslant 0 \ , \qquad (3.38)$$

where
$$C_{ij\nu o} \equiv C_{ij\nu}\Big|_{\tau=0} \quad (i, j = 1, \cdots n) \quad \nu \geqslant 0 \ ,$$

$$D_{ij\nu o} \equiv D_{ij\nu}\Big|_{\tau=0} \quad (i, j = 1, \cdots n) \quad \nu \geqslant 0 .$$

If, for the sake of convenience and simplicity, we introduce the notation

$$\begin{aligned} f_{ij\,\nu+1} &\equiv C_{ij\nu o}/\nu! \quad (i, j = 1, \cdots n) \quad \nu \geqslant 0 \ , \\ g_{ij\,\nu+1} &\equiv D_{ij\nu o}/\nu! \quad (i, j = 1, \cdots n) \quad \nu \geqslant 0 \ , \end{aligned} \qquad (3.39)$$

we get

$$\underset{\sim}{r}_{io}^{(\nu)} = \sum_{j=1}^{n} \left(f_{ij\,\nu+1}\, \underset{\sim}{r}_{jo} + g_{ij\,\nu+1}\, \underset{\sim}{r}_{jo}' \right) \nu! \quad (i = 1, \cdots n) \quad \nu \geqslant 0 \ ,$$

or

$$\frac{\underset{\sim}{r}_{io}^{(\nu)}}{\nu!} = \sum_{j=1}^{n} \left(f_{ij\,\nu+1}\, \underset{\sim}{r}_{jo} + g_{ij\,\nu+1}\, \underset{\sim}{r}_{jo}' \right) \quad (i = 1, \cdots n) \quad \nu \geqslant 0 \ . \qquad (3.40)$$

Substitution of equation (3.40) in the power series (3.37) yields

$$\underset{\sim}{r}_{i} = \sum_{\nu=0}^{\infty} \left(\sum_{j=1}^{n} f_{ij\,\nu+1}\, \underset{\sim}{r}_{jo} + g_{ij\,\nu+1}\, \underset{\sim}{r}_{jo}' \right) \tau^{\nu} \quad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \ ,$$

or
$$\underset{\sim}{r}_{i} = \sum_{j=1}^{n} \sum_{\nu=1}^{\infty} \left(f_{ij\nu}\, \underset{\sim}{r}_{jo} + g_{ij\nu}\, \underset{\sim}{r}_{jo}' \right) \tau^{\nu-1} \quad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \ ,$$

or
$$\underset{\sim}{r}_{i} = \sum_{j=1}^{n} \left\{ \left(\sum_{\nu=1}^{\infty} f_{ij\nu}\, \tau^{\nu-1} \right) \underset{\sim}{r}_{jo} + \left(\sum_{\nu=1}^{\infty} g_{ij\nu}\, \tau^{\nu-1} \right) \underset{\sim}{r}_{jo}' \right\} \quad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) . \qquad (3.41)$$

It is also obvious from equations (3.35), (3.36) and (3.39) that

$$f_{ij1} = \delta_{ij} \qquad (i, j = 1, \cdots n) \, ,$$
$$g_{ij1} = 0 \qquad (i, j = 1, \cdots n) \, ,$$

and (3.42)

$$f_{ij2} = 0 \qquad (i, j = 1, \cdots n) \, ,$$
$$g_{ij2} = \delta_{ij} \qquad (i, j = 1, \cdots n) \, .$$

Equation (3.41), being derived from equation (3.37), is valid at every interior point of the open interval $(-\tau^*, \tau^*)$, since this is the interval of convergence of the series (3.37). On the other hand, the vector $\underset{\sim}{r_i}$ $(i = 1, \ldots n)$, in relation (3.41), has a finite value in this interval of time. Hence, the coefficients of the vectors $\underset{\sim}{r_{jo}}$ and $\underset{\sim}{r_{jo}}'$ $(j = 1, \cdots n)$, which are in fact the series $\sum_{\nu=1}^{\infty} f_{ij\nu} \tau^{\nu-1}$, $\sum_{\nu=1}^{\infty} g_{ij\nu} \tau^{\nu-1}$ $(i, j = 1, \cdots n)$, must have precise values at every point $\tau \in (-\tau^*, \tau^*)$. This means that the series converge within interval of convergence $(-\tau^*, \tau^*)$. Hence

$$\begin{aligned} \sum_{\nu=1}^{\infty} f_{ij\nu} \tau^{\nu-1} &= f_{ij} \quad (i, j = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \, , \\ \sum_{\nu=1}^{\infty} g_{ij\nu} \tau^{\nu-1} &= g_{ij} \quad (i, j = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \, . \end{aligned} \tag{3.43}$$

Using equations (3.43) equation (3.41) becomes

$$\underset{\sim}{r_i} = \sum_{j=1}^{n} \left(f_{ij} \underset{\sim}{r_{jo}} + g_{ij} \underset{\sim}{r_{jo}}' \right) \quad (i = 1, \ldots n) \quad \tau \in (-\tau^*, \tau^*) . \tag{3.44}$$

Since $\underset{\sim}{r_{jo}}$ and $\underset{\sim}{r_{jo}}'$ $(j = 1, \cdots n)$ are constant, it is possible to differentiate equation (3.44) once with respect to τ to obtain the velocity vector $\underset{\sim}{r_i}'$ $(i = 1, \ldots n)$ as

$$\underset{\sim}{r_i}' = \sum_{j=1}^{n} \left(f_{ij}' \underset{\sim}{r_{jo}} + g_{ij}' \underset{\sim}{r_{jo}}' \right) \quad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) . \tag{3.45}$$

One more differentiation gives the acceleration vector $\underset{\sim}{r}_i''\ (i=1,\cdots n)$ as:

$$\underset{\sim}{r}_i'' = \sum_{j=1}^{n}\left(f_{ij}''\underset{\sim}{r}_{j0} + g_{ij}''\underset{\sim}{r}_{j0}'\right) \quad (i=1,\cdots n) \quad \tau\in(-\tau^*, \tau^*) \;. \tag{3.46}$$

On the other hand equation (3.10) is valid for all the values of the modified time τ, so changing the notation in this equation, we have

$$\underset{\sim}{r}_i'' = \left[-(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{k=1\\k\neq i}}^{n} m_k\frac{1}{r_{ik}^3}\right]\underset{\sim}{r}_i + \sum_{\substack{k=1\\k\neq i}}^{n} m_k\left(\frac{1}{r_{ik}^3} - \frac{1}{r_k^3}\right)\underset{\sim}{r}_k \quad (i=1,\cdots n) \quad \tau\in(-\tau^*, \tau^*) \,, \tag{3.47}$$

or, writing this in a more general form, we obtain

$$\underset{\sim}{r}_i'' = \sum_{k=1}^{n} A_{ik}\underset{\sim}{r}_k \quad (i=1\ldots n) \quad \tau\in(-\tau^*, \tau^*) \,, \tag{3.48}$$

where

$$\begin{aligned} A_{ii} &\equiv -(1+m_i)\frac{1}{r_i^3} - \sum_{\substack{k=1\\k\neq i}}^{n} m_k\frac{1}{r_{ik}^3} \quad (i=1,\ldots n) \,, \\ A_{ik} &\equiv m_k\left(\frac{1}{r_{ik}^3} - \frac{1}{r_k^3}\right) \quad (i,k=1,\ldots n \quad i\neq k) \,. \end{aligned} \tag{3.49}$$

Making use of equation (3.44), namely

$$\underset{\sim}{r}_k = \sum_{j=1}^{n}\left(f_{kj}\underset{\sim}{r}_{j0} + g_{kj}\underset{\sim}{r}_{j0}'\right) \quad (k=1,\ldots n) \quad \tau\in(-\tau^*, \tau^*) \,,$$

equation (3.48) becomes

$$\underset{\sim}{r}_i'' = \sum_{k=1}^{n} A_{ik}\left(\sum_{j=1}^{n}\left(f_{kj}\underset{\sim}{r}_{j0} + g_{kj}\underset{\sim}{r}_{j0}'\right)\right) \quad (i=1,\ldots n) \quad \tau\in(-\tau^*, \tau^*) \,,$$

or $$\underset{\sim}{r}_i'' = \sum_{k=1}^{n}\sum_{j=1}^{n}\left(A_{ik}f_{kj}\underset{\sim}{r}_{j0} + A_{ik}g_{kj}\underset{\sim}{r}_{j0}'\right) \quad (i=1,\ldots n) \quad \tau\in(-\tau^*, \tau^*) \,,$$

$$\text{or} \quad \underset{\sim}{\Gamma}_i'' = \sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} A_{ik} f_{kj}\right)\underset{\sim}{\Gamma}_{j0} + \left(\sum_{k=1}^{n} A_{ik} g_{kj}\right)\underset{\sim}{\Gamma}_{j0}'\right] \quad (i=1,\cdots n) \quad \tau\in(-\tau^*,\tau^*). \quad (3.50)$$

Subtracting equation (3.46) from equation (3.50) we obtain

$$\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} A_{ik} f_{kj} - f_{ij}''\right)\underset{\sim}{\Gamma}_{j0} + \left(\sum_{k=1}^{n} A_{ik} g_{kj} - g_{ij}''\right)\underset{\sim}{\Gamma}_{j0}'\right] = 0 \quad (i=1,\cdot\cdot n) \quad \tau\in(-\tau^*,\tau^*). \quad (3.51)$$

Since equation (3.51) is valid for every value of the variable τ in the open interval $(-\tau^*, \tau^*)$, it can be written as

$$\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} A_{ik} f_{kj} - f_{ij}''\right)\underset{\sim}{\Gamma}_{j0} + \left(\sum_{k=1}^{n} A_{ik} g_{kj} - g_{ij}''\right)\underset{\sim}{\Gamma}_{j0}'\right] \equiv 0 \quad (i=1,\cdots n) \quad \tau\in(-\tau^*,\tau^*). \quad (3.52)$$

So, the left hand side is identical to zero in the interval $(-\tau^*, \tau^*)$, and $\underset{\sim}{\Gamma}_{j0} \neq 0$, $\underset{\sim}{\Gamma}_{j0}' \neq 0$ $(j = 1, \cdots n)$. It can easily be shown, that the expressions which appear as coefficients of $\underset{\sim}{\Gamma}_{j0}$ and $\underset{\sim}{\Gamma}_{j0}'$, in relation (3.52), must have the values zero in the interval $(-\tau^*, \tau^*)$. That is

$$\sum_{k=1}^{n} A_{ik} f_{kj} = f_{ij}'' \quad (i,j = 1,\cdots n) \quad \tau\in(-\tau^*,\tau^*), \quad (3.53)$$

$$\sum_{k=1}^{n} A_{ik} g_{kj} = g_{ij}'' \quad (i,j = 1,\cdots n) \quad \tau\in(-\tau^*,\tau^*). \quad (3.54)$$

A function represented by power series has, in its interval of convergence, derivatives of every order which may be obtained by means of term-by-term successive differentiations. Hence, we easily obtain, by using equations (3.43), the derivatives f_{ij}'' and g_{ij}'' $(i,j = 1,\cdots n)$

$$\begin{aligned} f_{ij}'' &= \sum_{\nu=3}^{\infty}(\nu-1)(\nu-2)\, f_{ij\nu}\, \tau^{\nu-3} \quad (i,j=1,\cdots n) \quad \tau\in(-\tau^*,\tau^*), \\ g_{ij}'' &= \sum_{\nu=3}^{\infty}(\nu-1)(\nu-2)\, g_{ij\nu}\, \tau^{\nu-3} \quad (i,j=1,\cdots n) \quad \tau\in(-\tau^*,\tau^*), \end{aligned} \quad (3.55)$$

or

$$f''_{ij} = \sum_{\nu=1}^{\infty} \nu(\nu+1) f_{ij\nu+2} \tau^{\nu-1} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*), \tag{3.56}$$

$$g''_{ij} = \sum_{\nu=1}^{\infty} \nu(\nu+1) g_{ij\,\nu+2} \tau^{\nu-1} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*).$$

Substitution of these equations and equations (3.43) in equations (3.53) and (3.54) gives

$$\sum_{K=1}^{n} \left(A_{iK} \sum_{\nu=1}^{\infty} f_{Kj\nu} \tau^{\nu-1} \right) = \sum_{\nu=1}^{\infty} f_{ij\nu+2} (\nu+1) \nu \tau^{\nu-1} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*), \tag{3.57}$$

$$\sum_{K=1}^{n} \left(A_{iK} \sum_{\nu=1}^{\infty} g_{Kj\nu} \tau^{\nu-1} \right) = \sum_{\nu=1}^{\infty} g_{ij\nu+2} (\nu+1) \nu \tau^{\nu-1} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*). \tag{3.58}$$

Since A_{iK} $(i,K=1,\cdots n)$ are functions of the variable τ, and we do not know the analytical expressions $A_{iK} = A_{iK}(\tau)$ $(i,K=1,\cdots n)$, we can not solve the systems of the differential equations (3.57) and (3.58). So, in the first approximation we proceed as follows: We consider the functions $A_{iK}(\tau)$ $(i\,K=1,\cdots n)$ as constant and equal to $A_{iK}(\tau_1)$ $(i,K=1,\cdots n)$, where τ_1 belongs to the interval $(-\tau^*, \tau^*)$, then according to the identity theorem for power series the coefficients of terms of the same power of τ must be equal. That is

$$\nu(\nu+1) f_{ij\,\nu+2} = \sum_{K=1}^{n} A_{iK} f_{Kj\nu} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*) \quad \nu \geqslant 1, \tag{3.59}$$

$$\nu(\nu+1) g_{ij\nu+2} = \sum_{K=1}^{n} A_{iK} g_{Kj\nu} \quad (i,j=1,\cdots n) \quad \tau \in (-\tau^*, \tau^*) \quad \nu \geqslant 1. \tag{3.60}$$

Analytically, for every $\nu=1$, $\nu=2$, $\nu=3 \cdots$ we have

$$f_{ij3}=\left(\sum_{k=1}^{n}A_{ik}f_{kj1}\right)/1\cdot 2\,, \qquad f_{ij4}=\left(\sum_{k=1}^{n}A_{ik}f_{kj2}\right)/2\cdot 3\,,$$

$$f_{ij5}=\left(\sum_{k=1}^{n}A_{ik}f_{kj3}\right)/3\cdot 4\,, \qquad f_{ij6}=\left(\sum_{k=1}^{n}A_{ik}f_{kj4}\right)/4\cdot 5\,, \tag{3.61}$$

$$\vdots \qquad\qquad \vdots$$

$$f_{ij2\mu+1}=\left(\sum_{k=1}^{n}A_{ik}f_{kj2\mu-1}\right)/(2\mu-1)2\mu\,, \qquad f_{ij\,2\mu+2}=\left(\sum_{k=1}^{n}A_{ik}f_{kj2\mu}\right)/2\mu\cdot(2\mu+1)\,,$$

$$\vdots \qquad\qquad \vdots$$

$$(i,j=1,\ldots n)$$

and

$$g_{ij3}=\left(\sum_{k=1}^{n}A_{ik}g_{kj1}\right)/1\cdot 2\,, \qquad g_{ij4}=\left(\sum_{k=1}^{n}A_{ik}g_{kj2}\right)/2\cdot 3\,,$$

$$g_{ij5}=\left(\sum_{k=1}^{n}A_{ik}g_{kj3}\right)/3\cdot 4\,, \qquad g_{ij6}=\left(\sum_{k=1}^{n}A_{ik}g_{kj4}\right)/4\cdot 5\,,$$

$$\vdots \qquad\qquad \vdots$$

$$g_{ij2\mu+1}=\left(\sum_{k=1}^{n}A_{ik}g_{kj\,2\mu-1}\right)/(2\mu-1)\cdot 2\mu\,, \qquad g_{ij\,2\mu+2}=\left(\sum_{k=1}^{n}A_{ik}g_{kj\,2\mu}\right)/2\mu\cdot(2\mu+1)\,, \tag{3.62}$$

$$\vdots \qquad\qquad \vdots$$

$$(i,j=1,\ldots n).$$

Since these relations correlate the coefficients of the series of order $2\mu+1$ and $2\mu+2$ with the coefficients of order $2\mu-1$ and 2μ correspondingly, this enables us to find all the coefficients of the series in terms of the first two.

Introducing equations (3.42) in equations (3.61) and (3.62) we obtain

$$f_{ij1}=\delta_{ij}\,, \qquad f_{ij2}=0\,,$$

$$f_{ij3}=A_{ij}/2\,, \qquad f_{ij4}=0\,, \tag{3.63}$$

$$f_{ij5}=\left(\sum_{k=1}^{n}A_{ik}A_{kj}\right)/2\cdot 3\cdot 4\,, \qquad f_{ij6}=0\,,$$

$$f_{ij7} = \left(\sum_{\kappa=1}^{n} \sum_{\lambda=1}^{n} A_{i\kappa} A_{\kappa\lambda} A_{\lambda j}\right) / 6! \, , \qquad f_{ij8} = 0 \, ,$$

$$\vdots \qquad\qquad \vdots$$

$$(i, j = 1, \ldots n) \, ,$$

and

$$g_{ij1} = 0 \, , \qquad g_{ij2} = \delta_{ij} \, ,$$

$$g_{ij3} = 0 \, , \qquad g_{ij4} = A_{ij}/2 \cdot 3 \, , \tag{3.64}$$

$$g_{ij5} = 0 \, , \qquad g_{ij6} = \left(\sum_{\kappa=1}^{n} A_{i\kappa} A_{\kappa j}\right) / 5! \, ,$$

$$g_{ij7} = 0 \, , \qquad g_{ij8} = \left(\sum_{\kappa=1}^{n} \sum_{\lambda=1}^{n} A_{i\kappa} A_{\kappa\lambda} A_{\lambda j}\right) / 7! \, ,$$

$$\vdots \qquad\qquad \vdots$$

$$(i, j = 1, \ldots n) \, .$$

Since the coefficients of the even order of the series f_{ij} $(i, j = 1, \cdots n)$ are equal to zero, these series have the form

$$f_{ij} = \sum_{\nu=1}^{\infty} f_{ij\,2\nu-1} \, \tau^{2\nu-2} \quad (i, j = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \, . \tag{3.65}$$

Similarly, since the coefficients of the odd order of the series g_{ij} $(i, j = 1, \cdots n)$ are equal to zero, these series have the form

$$g_{ij} = \sum_{\nu=1}^{\infty} g_{ij\,2\nu} \, \tau^{2\nu-1} \quad (i, j = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \, . \tag{3.66}$$

It has already been shown that the series f_{ij} and g_{ij}

$(i, j = 1, \cdots n)$ converge in the interval $(-\tau^*, \tau^*)$ (for the general case). Since in the first approximation the series were derived with the assumption that the functions $A_{ik}(\tau)$ $(i, k = 1, \cdots n)$ are constant in the interval $(-\tau^*, \tau^*)$, we have to prove the convergence of these modified series.

Let us take $A \equiv \max\left\{ \left| A_{ik}(\tau) \right| (i, k = 1, 2, \cdots n), \tau \in (-\tau^*, \tau^*) \right\}$. From equations (3.63) it is easily derived that

$$\begin{aligned}
&\left| f_{ij3}\, \tau^2 \right| \leqslant A \frac{|\tau^2|}{2!} \equiv \alpha_1 , \\
&\left| f_{ij5}\, \tau^4 \right| \leqslant A^2 n \frac{|\tau^4|}{4!} \equiv \alpha_2 , \\
&\left| f_{ij7}\, \tau^6 \right| \leqslant A^3 n^2 \frac{|\tau^6|}{6!} \equiv \alpha_3 , \qquad\qquad (3.67) \\
&\vdots \qquad\qquad (i, j = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) \\
&\left| f_{ij\,2\mu+1}\, \tau^{2\mu} \right| \leqslant A^{\mu} n^{\mu-1} \frac{|\tau^{2\mu}|}{(2\mu)!} \equiv \alpha_{\mu} , \\
&\vdots
\end{aligned}$$

According to d'Alembert's ratio test the series

$$\sum_{\mu=1}^{\infty} \frac{A^{\mu} n^{\mu-1} |\tau^{2\mu}|}{(2\mu)!} \equiv \sum_{\mu=1}^{\infty} \alpha_{\mu}$$

is convergent for every time $\tau \in (-\tau^*, \tau^*)$ because

$$\lim_{\mu \to \infty} \frac{\alpha_{\mu+1}}{\alpha_{\mu}} \equiv \lim_{\mu \to \infty} \frac{\dfrac{A^{\mu+1} n^{\mu} |\tau^{2\mu+2}|}{(2\mu+2)!}}{\dfrac{A^{\mu} n^{\mu-1} |\tau^{2\mu}|}{(2\mu)!}} = \lim_{\mu \to \infty} \frac{A n |\tau^2|}{(2\mu+1)(2\mu+2)} = 0 < 1 .$$

Since this series converges, the absolute convergence, and so the simple convergence of the series f_{ij} $(i, j = 1, \cdots n)$, follows directly from the relations (3.67).

An entirely analogous process shows that the series g_{ij} $(i, j = 1, 2, \cdots n)$ converges in the same interval of time $(-\tau^*, \tau^*)$.

Unfortunately, we have no analytical expressions for the f_{ij} and g_{ij} series where $n > 2$. Consequently, in practical applications we have to truncate the series after a finite number of terms.

An extensive investigation for different numbers of terms and different values of τ shows that these series converge rapidly for values of τ less than 0.2, provided that the number of terms is greater than 60.

The series can easily be made to converge more rapidly by changing the units of the masses and distances (and hence the values of the quantities A_{ik} $(i, k = 1, \cdots n)$), and therefore the values of the coefficients of the series.

The calculation of the velocities of the bodies is now an easy problem, and is obtained as follows: Given the position vectors of the bodies at two different instants and their corresponding times, say $\underset{\sim}{r}_{i1}$, $\underset{\sim}{r}_{i2}$ $(i = 1, \cdots n)$, t_1, t_2 we can take

$$t_0 \equiv t_1 \quad , \quad t \equiv t_2 \quad ,$$
$$\underset{\sim}{r}_{i0} \equiv \underset{\sim}{r}_{i1} \quad , \quad \underset{\sim}{r}_i \equiv \underset{\sim}{r}_{i2} \quad (i = 1, \cdots n) .$$

Then the value of the modified time $\tau = k(t - t_0)$ is calculated. With the presupposition that this value is included in the interval of convergence of the series (3.37), equations (3.49), (3.63) and (3.64) are used to calculate the values of the series f_{ij}, g_{ij} $(i, j = 1, 2, \cdots n)$ Then the solution of the system

$$\underset{\sim}{r}_i = \sum_{j=1}^{n} \left(f_{ij}\, \underset{\sim}{r}_{j0} + g_{ij}\, \underset{\sim}{r}_{j0}' \right) \qquad (i = 1, \ldots n)$$

gives the velocity vectors of the bodies, $\underset{\sim}{r}_{j0}'$ $(j = 1, \cdots n)$, with respect to the known position vectors $\underset{\sim}{r}_{j0}$ $(j = 1, \cdots n)$ and $\underset{\sim}{r}_i$ $(i = 1, \cdots n)$.

Having found the velocity vectors as functions of the position vectors of the bodies, we now return to the differential equations (3.13) which describe the relative motion of the bodies.

We will next develop a method for the calculation of the coefficients of the power series (3.37), which is the solution of the system (3.13).

If we resolve the vector series (3.37) into components series then the solution has the form

$$\begin{aligned} x_i &= \sum_{\nu=1}^{\infty} x_{i\nu}\, \tau^{\nu-1} , \\ y_i &= \sum_{\nu=1}^{\infty} y_{i\nu}\, \tau^{\nu-1} , \\ z_i &= \sum_{\nu=1}^{\infty} z_{i\nu}\, \tau^{\nu-1} , \end{aligned} \qquad (3.68)$$

$$(i = 1, \cdots n) \qquad \tau \in (-\tau^*, \tau^*) ,$$

where

$$\underset{\sim}{r}_{i0}^{(\nu)} / \nu! \equiv \left(x_{i\nu+1}, y_{i\nu+1}, z_{i\nu+1} \right) \quad (i = 1, \cdots n) \quad \nu \geq 0 .$$

Note that the above series are initiated at $\nu = 1$, as this change in notation will make subsequent numerical calculations (with the help of computer) feasible.

The solution (3.68) must be subjected to certain initial conditions. In order to find them the series (3.68) are considered at time $\tau = 0$. It is obvious that

$$x_{i_{\tau=0}} = x_{i1} ,$$
$$y_{i_{\tau=0}} = y_{i1} , \tag{3.69}$$
$$z_{i_{\tau=0}} = z_{i1} ,$$
$$(i = 1, \cdots n) .$$

Hence, the co-ordinates of the positions of the bodies, at time $\tau = 0$ are equal to the first terms of the power series (3.68).

Differentiating once equation (3.68) with respect to τ, the following is obtained

$$x_i' = u_i = \sum_{\nu=2}^{\infty} x_{i\nu} (\nu-1) \tau^{\nu-2} ,$$
$$y_i' = v_i = \sum_{\nu=2}^{\infty} y_{i\nu} (\nu-1) \tau^{\nu-2} , \tag{3.70}$$
$$z_i' = w_i = \sum_{\nu=2}^{\infty} z_{i\nu} (\nu-1) \tau^{\nu-2} ,$$
$$(i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*) .$$

At the point $\tau = 0$ equations (3.70) give

$$x'_{i_{\tau=0}} = u_{i_{\tau=0}} = x_{i2} ,$$
$$y'_{i_{\tau=0}} = v_{i_{\tau=0}} = y_{i2} , \tag{3.71}$$
$$z'_{i_{\tau=0}} = w_{i_{\tau=0}} = z_{i2} ,$$
$$(i = 1, \cdots n) .$$

Therefore, the components of the velocities of the bodies at the point $\tau = 0$ are equal to the coefficients of the second terms of the series (3.68).

Equations (3.69) and (3.71) give the initial conditions to which the power series solution must be subjected.

The next step in this process is to obtain expansions in power series for the other variables of the equations of motion.

We work in three-dimensional Euclidean space, hence the relation

$$r_i^2 = x_i^2 + y_i^2 + z_i^2 \qquad (i = 1, \cdots n), \tag{3.72}$$

is valid for every value of the time τ. Since the functions $x_i = x_i(\tau)$, $y_i = y_i(\tau)$ and $z_i = z_i(\tau)$ $(i = 1, \cdots n)$ are represented by power series in the interval $(-\tau^*, \tau^*)$, and the function $r_i = r_i(\tau)$ $(i = 1, \cdots n)$ has derivatives of all orders, according to the theory of infinite series, the relation (3.72) assures us of a power series expansion of the form

$$r_i = \sum_{\nu=1}^{\infty} r_{i\nu} \tau^{\nu-1} \qquad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*). \tag{3.73}$$

Differentiation of equations (3.72) and (3.73) gives

$$\begin{aligned} r_i r_i' &= x_i x_i' + y_i y_i' + z_i z_i' \qquad (i = 1, \cdots n), \\ r_i' &= \sum_{\nu=2}^{\infty} r_{i\nu} (\nu-1) \tau^{\nu-2} \qquad (i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*). \end{aligned} \tag{3.74}$$

By introducing equations (3.68), (3.70), (3.73) and the second of the equations (3.74) in the first of the equations (3.74), the following is obtained

$$\begin{aligned} \Big(\sum_{\nu=1}^{\infty} r_{i\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} r_{i\nu} (\nu-1) \tau^{\nu-2}\Big) &= \Big(\sum_{\nu=1}^{\infty} x_{i\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} x_{i\nu} (\nu-1) \tau^{\nu-2}\Big) + \\ + \Big(\sum_{\nu=1}^{\infty} y_{i\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} y_{i\nu} (\nu-1) \tau^{\nu-2}\Big) &+ \Big(\sum_{\nu=1}^{\infty} z_{i\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} z_{i\nu} (\nu-1) \tau^{\nu-2}\Big) \\ &(i = 1, \cdots n) \quad \tau \in (-\tau^*, \tau^*), \end{aligned}$$

$$\text{or}\quad \sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} r_{i\kappa}\, r_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1} = \sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} x_{i\kappa}\, x_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1} + $$
$$+\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} y_{i\kappa}\, y_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1} + \qquad (3.75)$$
$$+\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} z_{i\kappa}\, z_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1}$$
$$(i=1,\dots n)\qquad \tau\in(-\tau^*,\tau^*)\ .$$

By equating the coefficients of terms of the same power of τ, it follows that

$$\sum_{\kappa=1}^{\nu} r_{i\kappa}\, r_{i\,\nu+2-\kappa}(\nu+1-\kappa) = \sum_{\kappa=1}^{\nu} x_{i\kappa}\, x_{i\,\nu+2-\kappa}(\nu+1-\kappa) + $$
$$+\sum_{\kappa=1}^{\nu} y_{i\kappa}\, y_{i\nu+2-\kappa}(\nu+1-\kappa) + $$
$$+\sum_{\kappa=1}^{\nu} z_{i\kappa}\, z_{i\nu+2-\kappa}(\nu+1-\kappa)\quad (i=1,\dots n)\quad \nu\geqslant 1\,,$$

$$\text{or}\ \sum_{\kappa=1}^{\nu} r_{i\kappa}\, r_{i\nu+2-\kappa}(\nu+1-\kappa) = \sum_{\kappa=1}^{\nu}(x_{i\kappa}\, x_{i\nu+2-\kappa} + y_{i\kappa}\, y_{i\nu+2-\kappa} + z_{i\kappa}\, z_{i\nu+2-\kappa})(\nu+1-\kappa)\quad (i=1,\dots n)\ \nu\geqslant 1,$$

or

$$\nu\cdot r_{i1}\, r_{i\nu+1} + \sum_{\kappa=2}^{\nu} r_{i\kappa}\, r_{i\nu+2-\kappa}(\nu+1-\kappa) =$$
$$=\sum_{\kappa=1}^{\nu}(x_{i\kappa}\, x_{i\nu+2-\kappa} + y_{i\kappa}\, y_{i\nu+2-\kappa} + z_{i\kappa}\, z_{i\nu+2-\kappa})(\nu+1-\kappa)\quad (i=1,\dots n)\quad \nu\geqslant 1\ .$$

hence

$$r_{i\nu+1} = \frac{1}{r_{i1}\nu}\Big[-\sum_{\kappa=2}^{\nu} r_{i\kappa}\, r_{i\nu+2-\kappa}(\nu+1-\kappa) + \sum_{\kappa=1}^{\nu}(x_{i\kappa}\, x_{i\nu+2-\kappa} + y_{i\kappa}\, y_{i\nu+2-\kappa} + z_{i\kappa}\, z_{i\nu+2-\kappa})(\nu+1-\kappa)\Big]\qquad (3.76)$$
$$(i=1,\dots n)\qquad \nu\geqslant 1\ .$$

This relation gives the coefficients of every order (greater than two) of the series (3.73) in terms of the lower order coefficients of the same series, and in terms of the lower or equal order coefficients of the series (3.68).

In order to find the first order coefficient, r_{i1} $(i=1,\dots n)$, the series (3.73) is considered at the point $\tau=0$ and gives

$$r_{i\,\tau=0}=r_{i1} \qquad (i=1,2,\dots n)\,. \tag{3.77}$$

On the other hand, equation (3.72), at the point $\tau=0$ gives

$$r_{i\,\tau=0}=\left(x^2_{i\,\tau=0}+y^2_{i\,\tau=0}+z^2_{i\,\tau=0}\right)^{1/2} \qquad (i=1,\dots n)\,, \tag{3.78}$$

and substituting equations (3.69) and (3.77), equation (3.78) yields

$$r_{i1}=\left(x^2_{i1}+y^2_{i1}+z^2_{i1}\right)^{1/2} \qquad (i=1,\cdots n)\,. \tag{3.79}$$

As in the case of the function $r_i=r_i(\tau)$ $(i=1,\cdots n)$, it can be similarly proved that the function $r_{ij}=r_{ij}(\tau)$ $(i,j=1,\cdots n\;\; i\neq j)$ possesses a power series expansion, in the interval $(-\tau^*,\tau^*)$, that is

$$r_{ij}=\sum_{\nu=1}^{\infty} r_{ij\nu}\,\tau^{\nu-1} \qquad (i,j=1,\dots n\;\; i\neq j) \quad \tau\in(-\tau^*,\tau^*)\,. \tag{3.80}$$

The equation

$$r_{ij}^2=(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2 \qquad (i,j=1,\dots n\;\; i\neq j)\,, \tag{3.81}$$

is valid for every value of the time τ.

Differentiation of equations (3.80) and (3.81) gives

$$r_{ij}\, r_{ij}' = (x_i - x_j)(x_i' - x_j') + (y_i - y_j)(y_i' - y_j') + (z_i - z_j)(z_i' - z_j') \quad (i,j = 1,\dots n \;\; i \neq j)\,,$$

$$r_{ij}' = \sum_{\nu=2}^{\infty} r_{ij\nu}\,(\nu-1)\,\tau^{\nu-2} \quad (i,j = 1,\dots n \;\; i \neq j) \quad \tau \in (-\tau^*, \tau^*)\,. \tag{3.82}$$

Making use of the equations (3.68), (3.70) and the second of the equations (3.82), the first of the equations (3.82) becomes

$$\Big(\sum_{\nu=1}^{\infty} r_{ij\nu}\,\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} r_{ij\nu}\,(\nu-1)\,\tau^{\nu-2}\Big) = \Big(\sum_{\nu=1}^{\infty} x_{i\nu}\tau^{\nu-1} - \sum_{\nu=1}^{\infty} x_{j\nu}\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} x_{i\nu}(\nu-1)\tau^{\nu-2} - \sum_{\nu=2}^{\infty} x_{j\nu}(\nu-1)\tau^{\nu-2}\Big)$$
$$+ \Big(\sum_{\nu=1}^{\infty} y_{i\nu}\tau^{\nu-1} - \sum_{\nu=1}^{\infty} y_{j\nu}\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} y_{i\nu}(\nu-1)\tau^{\nu-2} - \sum_{\nu=2}^{\infty} y_{j\nu}(\nu-1)\tau^{\nu-2}\Big) +$$
$$+ \Big(\sum_{\nu=1}^{\infty} z_{i\nu}\tau^{\nu-1} - \sum_{\nu=1}^{\infty} z_{j\nu}\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} z_{i\nu}(\nu-1)\tau^{\nu-2} - \sum_{\nu=2}^{\infty} z_{j\nu}(\nu-1)\tau^{\nu-2}\Big)$$
$$(i,j = 1,\dots n \;\; i \neq j) \quad \tau \in (-\tau^*, \tau^*)\,,$$

or

$$\Big(\sum_{\nu=1}^{\infty} r_{ij\nu}\,\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} r_{ij\nu}\,(\nu-1)\,\tau^{\nu-2}\Big) = \Big(\sum_{\nu=1}^{\infty}(x_{i\nu} - x_{j\nu})\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty}(\nu-1)(x_{i\nu} - x_{j\nu})\tau^{\nu-2}\Big) +$$
$$+ \Big(\sum_{\nu=1}^{\infty}(y_{i\nu} - y_{j\nu})\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty}(\nu-1)(y_{i\nu} - y_{j\nu})\tau^{\nu-2}\Big) +$$
$$+ \Big(\sum_{\nu=1}^{\infty}(z_{i\nu} - z_{j\nu})\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty}(\nu-1)(z_{i\nu} - z_{j\nu})\tau^{\nu-2}\Big)$$
$$(i,j = 1,\dots n \;\; i \neq j) \quad \tau \in (-\tau^*, \tau^*)\,,$$

therefore

$$\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} r_{ij\kappa}\, r_{ij\,\nu+2-\kappa}\,(\nu+1-\kappa)\Big)\tau^{\nu-1} = \sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu}(x_{i\kappa} - x_{j\kappa})(x_{i\,\nu+2-\kappa} - x_{j\,\nu+2-\kappa})(\nu+1-\kappa)\Big)\tau^{\nu-1} +$$
$$+ \sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu}(y_{i\kappa} - y_{j\kappa})(y_{i\,\nu+2-\kappa} - y_{j\,\nu+2-\kappa})(\nu+1-\kappa)\Big)\tau^{\nu-1} +$$
$$+ \sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu}(z_{i\kappa} - z_{j\kappa})(z_{i\,\nu+2-\kappa} - z_{j\,\nu+2-\kappa})(\nu+1-\kappa)\Big)\tau^{\nu-1}$$
$$(i,j = 1,\dots n \;\; i \neq j) \quad \tau \in (-\tau^*, \tau^*)\,.$$

Equating the coefficients of terms of the same power of τ the following is obtained

$$\sum_{K=1}^{\nu} r_{ijK} r_{ij\,\nu+2-K}(\nu+1-K) = \sum_{K=1}^{\nu}(x_{iK}-x_{jK})(x_{i\,\nu+2-K}-x_{j\,\nu+2-K})(\nu+1-K) +$$
$$+\sum_{K=1}^{\nu}(y_{iK}-y_{jK})(y_{i\,\nu+2-K}-y_{j\,\nu+2-K})(\nu+1-K)+$$
$$+\sum_{K=1}^{\nu}(z_{iK}-z_{jK})(z_{i\,\nu+2-K}-z_{j\,\nu+2-K})(\nu+1-K)$$
$$(i,j=1,\ldots n \quad i\neq j) \qquad \nu \geqslant 1 \quad ,$$

or

$$\nu\, r_{ij1}\, r_{ij\,\nu+1} + \sum_{K=2}^{\nu} r_{ijK} r_{ij\,\nu+2-K}(\nu+1-K) =$$
$$=\sum_{K=1}^{\nu}(\nu+1-K)\Big[(x_{iK}-x_{jK})(x_{i\,\nu+2-K}-x_{j\,\nu+2-K})+$$
$$+(y_{iK}-y_{jK})(y_{i\,\nu+2-K}-y_{j\,\nu+2-K})+(z_{iK}-z_{jK})(z_{i\,\nu+2-K}-z_{j\,\nu+2-K})\Big]$$
$$(i,j=1,\ldots n \quad i\neq j) \quad \nu\geqslant 1 \quad .$$

Solution of this equation with respect to $r_{ij\,\nu+1}$ gives:

$$r_{ij\,\nu+1} = \Big\{-\sum_{K=2}^{\nu} r_{ijK} r_{ij\,\nu+2-K}(\nu+1-K)+\sum_{K=1}^{\nu}(\nu+1-K)\Big[(x_{iK}-x_{jK})(x_{i\,\nu+2-K}-x_{j\,\nu+2-K})+$$
$$+(y_{iK}-y_{jK})(y_{i\,\nu+2-K}-y_{j\,\nu+2-K})+(z_{iK}-z_{jK})(z_{i\,\nu+2-K}-z_{j\,\nu+2-K})\Big]\Big\}\Big/\nu\, r_{ij1} \quad (3.83)$$
$$(i,j=1,\ldots n \quad i\neq j) \qquad \nu\geqslant 1 \quad ,$$

and since this relation gives the coefficients $r_{ij\nu}$ for $\nu \geqslant 2$ the coefficient r_{ij1} $(i,j=1,\ldots n \; i\neq j)$ must be found by a different method. The power series (3.80), at the point $\tau=0$, reduces to the relation

$$r_{ij\,\tau=0} = r_{ij1} \qquad (i, j=1, \ldots n \quad i\neq j) , \tag{3.84}$$

while, on the other hand, equation (3.81) at the point $\tau=0$, becomes

$$r_{ij\,\tau=0} = \left[(x_{i\,\tau=0}-x_{j\,\tau=0})^2+(y_{i\,\tau=0}-y_{j\,\tau=0})^2+(z_{i\,\tau=0}-z_{j\,\tau=0})^2\right]^{1/2} \tag{3.85}$$
$$(i, j=1, \ldots n \quad i\neq j) ,$$

which, by introduction of equations (3.69) and (3.84), becomes

$$r_{ij1} = \left[(x_{i1}-x_{j1})^2+(y_{i1}-y_{j1})^2+(z_{i1}-z_{j1})^2\right]^{1/2} \quad (i, j=1, \ldots n \quad i\neq j) . \tag{3.86}$$

Now, because the function $r_i = r_i(\tau) \quad (i=1, \ldots n)$ can be expressed as a power series in the open interval $(-\tau^*, \tau^*)$, the same will be valid for the function $R_i = r_i^{-3}(\tau) \quad (i=1, \ldots n)$ in the same interval of time. Hence, it can be written that

$$r_i^{-3} = \sum_{\nu=1}^{\infty} R_{i\nu}\tau^{\nu-1} \qquad (i=1, \ldots n) \quad \tau\in(-\tau^*, \tau^*) . \tag{3.87}$$

Differentiating $r_i^{-3} \quad (i=1, \ldots n)$ with respect to τ gives

$$(r_i^{-3})' = \sum_{\nu=2}^{\infty} R_{i\nu}(\nu-1)\tau^{\nu-2} \qquad (i=1, \ldots n) \quad \tau\in(-\tau^*, \tau^*) , \tag{3.88}$$

also

$$(r_i^{-3})' = -3\, r_i^{-4}\, r_i' \qquad (i=1, \ldots n) \quad \tau\in(-\tau^*, \tau^*) ,$$

or

$$(r_i^{-3})'\, r_i = -3\, r_i^{-3}\, r_i' \qquad (i=1, \ldots n) \quad \tau\in(-\tau^*, \tau^*) . \tag{3.89}$$

The last equation, by introduction of equations (3.73), (3.74), (3.87) and (3.88), becomes

$$\Big(\sum_{\nu=1}^{\infty} r_{i\nu}\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} R_{i\nu}(\nu-1)\tau^{\nu-2}\Big)=-3\Big(\sum_{\nu=1}^{\infty} R_{i\nu}\tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} r_{i\nu}(\nu-1)\tau^{\nu-2}\Big) \quad ,$$

or $$\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} r_{i\kappa}R_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1}=-3\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} R_{i\kappa}r_{i\,\nu+2-\kappa}(\nu+1-\kappa)\Big)\tau^{\nu-1}$$

$$(i=1,\ldots n) \quad \tau\in(-\tau^*,\tau^*) .$$

By equating the coefficients of terms of the same power of τ it follows that

$$\sum_{\kappa=1}^{\nu} r_{i\kappa}R_{i\,\nu+2-\kappa}(\nu+1-\kappa)=-3\sum_{\kappa=1}^{\nu} R_{i\kappa}r_{i\,\nu+2-\kappa}(\nu+1-\kappa) \quad (i=1,\ldots n) \quad \nu\geqslant 1 ,$$

or $$\nu r_{i1}R_{i\,\nu+1}+\sum_{\kappa=2}^{\nu} r_{i\kappa}R_{i\,\nu+2-\kappa}(\nu+1-\kappa)=-3\sum_{\kappa=1}^{\nu} R_{i\kappa}r_{i\,\nu+2-\kappa}(\nu+1-\kappa)$$

$$(i=1,\ldots n) \quad \nu\geqslant 1 ,$$

and the solution with respect to $R_{i\,\nu+1}$ yields the result

$$R_{i\,\nu+1}=\left[-\sum_{\kappa=2}^{\nu}(\nu+1-\kappa)r_{i\kappa}R_{i\,\nu+2-\kappa}-3\sum_{\kappa=1}^{\nu} R_{i\kappa}r_{i\,\nu+2-\kappa}(\nu+1-\kappa)\right]\Big/\nu r_{i1} \tag{3.90}$$

$$(i=1,\ldots n) \quad \nu\geqslant 1 .$$

The coefficient R_{i1} $(i=1,\ldots n)$ is obtained as follows: From the power series expansion of r_i^{-3} $(i=1,\ldots n)$, at time $\tau=0$, it is found that

$$r_{i\,\tau=0}^{-3}=R_{i1} \quad (i=1,\ldots n) ,$$

which, with equations (3.77) and (3.79), gives

$$R_{i1}=(x_{i1}^2+y_{i1}^2+z_{i1}^2)^{-3/2} \quad (i=1,\ldots n) . \tag{3.91}$$

Now, it can be easily proved that the function $r_{ij}^{-3} = R_{ij}(\tau)$ $(i, j = 1, \ldots n \;\; i \neq j)$ possesses a power series expansion in the interval $(-\tau^*, \tau^*)$. Consider, therefore

$$r_{ij}^{-3} = \sum_{\nu=1}^{\infty} R_{ij\nu} \tau^{\nu-1} \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*), \tag{3.92}$$

then differentiating r_{ij}^{-3} $(i, j = 1, \ldots n \;\; i \neq j)$, with respect to τ, gives

$$(r_{ij}^{-3})' = \sum_{\nu=2}^{\infty} R_{ij\nu} (\nu-1) \tau^{\nu-2} \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*), \tag{3.93}$$

also

$$(r_{ij}^{-3})' = -3 r_{ij}^{-4} r_{ij}' \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*),$$

or

$$r_{ij} (r_{ij}^{-3})' = -3 r_{ij}^{-3} r_{ij}' \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*). \tag{3.94}$$

Equation (3.94), by introduction of equations (3.80), (3.82), (3.92) and (3.93), becomes

$$\Big(\sum_{\nu=1}^{\infty} r_{ij\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} R_{ij\nu} (\nu-1) \tau^{\nu-2}\Big) =$$

$$= -3 \Big(\sum_{\nu=1}^{\infty} R_{ij\nu} \tau^{\nu-1}\Big)\Big(\sum_{\nu=2}^{\infty} r_{ij\nu} (\nu-1) \tau^{\nu-2}\Big) \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*),$$

or

$$\sum_{\nu=1}^{\infty} \Big(\sum_{\kappa=1}^{\nu} (\nu+1-\kappa) r_{ij\kappa} R_{ij\,\nu+2-\kappa}\Big) \tau^{\nu-1} =$$

$$= -3 \sum_{\nu=1}^{\infty} \Big(\sum_{\kappa=1}^{\nu} (\nu+1-\kappa) R_{ij\kappa} r_{ij\,\nu+2-\kappa}\Big) \tau^{\nu-1} \quad (i, j = 1, \ldots n \;\; i \neq j) \;\; \tau \in (-\tau^*, \tau^*).$$

Making use of the identity theorem for power series we obtain

$$\sum_{\kappa=1}^{\nu}(\nu+1-\kappa)\, r_{ij\kappa}\, R_{ij\,\nu+2-\kappa} =$$

$$= -3\sum_{\kappa=1}^{\nu}(\nu+1-\kappa) R_{ij\kappa}\, r_{ij\,\nu+2-\kappa} \qquad (i,j=1,\ldots n \;\; i\neq j) \qquad \nu \geqslant 1 \quad ,$$

or

$$\nu\, r_{ij1} R_{ij\,\nu+1} + \sum_{\kappa=2}^{\nu}(\nu+1-\kappa)\, r_{ij\kappa} R_{ij\,\nu+2-\kappa} =$$

$$= -3\sum_{\kappa=1}^{\nu}(\nu+1-\kappa) R_{ij\kappa}\, r_{ij\,\nu+2-\kappa} \qquad (i,j=1,\ldots n \;\; i\neq j) \qquad \nu\geqslant 1 \quad .$$

Solution with respect to $R_{ij\,\nu+1}$ $(i,j=1,\ldots n \;\; i\neq j)$ yields the result

$$R_{ij\,\nu+1} = \left[-\sum_{\kappa=2}^{\nu}(\nu+1-\kappa)\, r_{ij\kappa} R_{ij\,\nu+2-\kappa} - 3\sum_{\kappa=1}^{\nu}(\nu+1-\kappa) R_{ij\kappa}\, r_{ij\,\nu+2-\kappa}\right] \Big/ \nu\, r_{ij1} \qquad (3.95)$$

$$(i,j=1,\ldots n \;\; i\neq j) \qquad \nu\geqslant 1 \; .$$

In order to find the first order coefficient R_{ij1} $(i,j=1, 2,\ldots n \;\; i\neq j)$, the series (3.92) is considered at the point $\tau=0$. Evidently

$$R_{ij1} = r_{ij}^{-3}{}_{\tau=0} \qquad (i,j=1,\ldots n \;\; i\neq j) \, ,$$

which, with equations (3.84) and (3.86), becomes

$$R_{ij1} = \left[(x_{i1}-x_{j1})^2+(y_{i1}-y_{j1})^2+(z_{i1}-z_{j1})^2\right]^{-3/2} \quad (i,j=1,\ldots n \;\; i\neq j) \, . \qquad (3.96)$$

The co-ordinates of the position vectors of the bodies, being considered as functions of the independent variable τ can be expanded in series (3.68). Hence, their first-order derivatives have power series expansions with the same interval of convergence $(-\tau^*, \tau^*)$, which are

$$x_i' \equiv u_i = \sum_{\nu=1}^{\infty} u_{i\nu}\, \tau^{\nu-1} \quad ,$$

$$y_i' \equiv v_i = \sum_{\nu=1}^{\infty} v_{i\nu}\, \tau^{\nu-1} \quad , \qquad (3.97)$$

$$Z_i' \equiv w_i = \sum_{\nu=1}^{\infty} w_{i\nu} \tau^{\nu-1}$$

$$(i = 1, \ldots n) \qquad \tau \in (-\tau^*, \tau^*) \quad .$$

Using expansions (3.70), the following equations are easily obtained

$$\sum_{\nu=2}^{\infty} x_{i\nu} (\nu-1) \tau^{\nu-2} = \sum_{\nu=1}^{\infty} u_{i\nu} \tau^{\nu-1} \quad ,$$

$$\sum_{\nu=2}^{\infty} y_{i\nu} (\nu-1) \tau^{\nu-2} = \sum_{\nu=1}^{\infty} v_{i\nu} \tau^{\nu-1} \quad ,$$

$$\sum_{\nu=2}^{\infty} z_{i\nu} (\nu-1) \tau^{\nu-2} = \sum_{\nu=1}^{\infty} w_{i\nu} \tau^{\nu-1} \quad ,$$

$$(i = 1, \ldots n) \qquad \tau \in (-\tau^*, \tau^*) \quad .$$

The above three equations hold for every value of τ in the common interval of convergence of the series, hence the identity theorem for power series secures that the coefficients of terms of the same power of τ are equal. That is

$$\begin{aligned} \nu x_{i\nu+1} &= u_{i\nu} \qquad \text{or} \qquad x_{i\nu+1} = u_{i\nu}/\nu \ , \\ \nu y_{i\nu+1} &= v_{i\nu} \qquad \text{or} \qquad y_{i\nu+1} = v_{i\nu}/\nu \ , \\ \nu z_{i\nu+1} &= w_{i\nu} \qquad \text{or} \qquad z_{i\nu+1} = w_{i\nu}/\nu \ , \end{aligned} \tag{3.98}$$

$$(i = 1, \ldots n) \qquad \nu \geqslant 1 \quad .$$

The equations of motion give the last three of the recursion formulae. Equation (3.10) is equivalent to the following equations for the co-ordinates x_i, y_i, z_i $(i = 1, \ldots n)$

$$x_i'' = u_i' = -(1+m_i)\frac{x_i}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{x_i - x_j}{r_{ij}^3} + \frac{x_j}{r_j^3} \right) ,$$

$$y_i'' = v' = -(1+m_i)\frac{y_i}{r_i^3} - \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{y_i - y_j}{r_{ij}^3} + \frac{y_j}{r_j^3} \right) , \tag{3.99}$$

$$Z_i'' = W_i' = -(1+m_i)\frac{Z_i}{r_i^3} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left(\frac{Z_i - Z_j}{r_{ij}^3} + \frac{Z_j}{r_j^3}\right),$$

$$(i = 1, \ldots n) .$$

Introduction of the power series expansions (3.68), (3.87), (3.92), (3.97) in equations (3.99) gives

$$\sum_{\nu=2}^{\infty} u_{i\nu}(\nu-1)\tau^{\nu-2} = -(1+m_i)\left(\sum_{\nu=1}^{\infty} x_{i\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{i\nu}\tau^{\nu-1}\right) -$$
$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left\{\left(\sum_{\nu=1}^{\infty}(x_{i\nu}-x_{j\nu})\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{ij\nu}\tau^{\nu-1}\right) + \left(\sum_{\nu=1}^{\infty} x_{j\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{j\nu}\tau^{\nu-1}\right)\right\},$$

$$\sum_{\nu=2}^{\infty} V_{i\nu}(\nu-1)\tau^{\nu-2} = -(1+m_i)\left(\sum_{\nu=1}^{\infty} y_{i\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{i\nu}\tau^{\nu-1}\right) -$$
$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left\{\left(\sum_{\nu=1}^{\infty}(y_{i\nu}-y_{j\nu})\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{ij\nu}\tau^{\nu-1}\right) + \left(\sum_{\nu=1}^{\infty} y_{j\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{j\nu}\tau^{\nu-1}\right)\right\},$$

(3.100)

$$\sum_{\nu=2}^{\infty} W_{i\nu}\ (\nu-1)\,\tau^{\nu-2} =$$
$$= -(1+m_i)\left(\sum_{\nu=1}^{\infty} z_{i\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{i\nu}\tau^{\nu-1}\right) -$$

$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left\{\left(\sum_{\nu=1}^{\infty}(z_{i\nu}-z_{j\nu})\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{ij\nu}\tau^{\nu-1}\right) + \left(\sum_{\nu=1}^{\infty} z_{j\nu}\tau^{\nu-1}\right)\left(\sum_{\nu=1}^{\infty} R_{j\nu}\tau^{\nu-1}\right)\right\},$$

$$(i = 1, \ldots n) \quad \tau \in (-\tau^*, \tau^*) .$$

The derivatives u_i', v_i' and w_i' were obtained by differentiation of the power series (3.97) term-by-term. Equations (3.100) can be written as follows:

$$\sum_{\nu=1}^{\infty} u_{i\nu+1}\, \nu\, \tau^{\nu-1} =$$
$$= -(1+m_i)\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} x_{i\kappa} R_{i\nu+1-\kappa}\Big)\tau^{\nu-1} -$$
$$-\sum_{\nu=1}^{\infty}\Bigg\{\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big[\Big(\sum_{\kappa=1}^{\nu}(x_{i\kappa}-x_{j\kappa})R_{ij\nu+1-\kappa}\Big)+\Big(\sum_{\kappa=1}^{\nu}x_{j\kappa}R_{j\nu+1-\kappa}\Big)\Big]\Bigg\}\tau^{\nu-1},$$

$$\sum_{\nu=1}^{\infty} v_{i\nu+1}\, \nu\, \tau^{\nu-1} =$$
$$= -(1+m_i)\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} y_{i\kappa} R_{i\nu+1-\kappa}\Big)\tau^{\nu-1} -$$
$$-\sum_{\nu=1}^{\infty}\Bigg\{\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big[\Big(\sum_{\kappa=1}^{\nu}(y_{i\kappa}-y_{j\kappa})R_{ij\nu+1-\kappa}\Big)+\Big(\sum_{\kappa=1}^{\nu}y_{j\kappa}R_{j\nu+1-\kappa}\Big)\Big]\Bigg\}\tau^{\nu-1}, \qquad (3.101)$$

$$\sum_{\nu=1}^{\infty} w_{i\nu+1}\, \nu\, \tau^{\nu-1} =$$
$$= -(1+m_i)\sum_{\nu=1}^{\infty}\Big(\sum_{\kappa=1}^{\nu} z_{i\kappa} R_{i\nu+1-\kappa}\Big)\tau^{\nu-1} -$$
$$-\sum_{\nu=1}^{\infty}\Bigg\{\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big[\Big(\sum_{\kappa=1}^{\nu}(z_{i\kappa}-z_{j\kappa})R_{ij\nu+1-\kappa}\Big)+\Big(\sum_{\kappa=1}^{\nu}z_{j\kappa}R_{j\nu+1-\kappa}\Big)\Big]\Bigg\}\tau^{\nu-1},$$

$$(i=1,\ldots n) \quad \tau\in(-\tau^*,\tau^*).$$

Equating the coefficients of terms of the same power of τ we obtain

$$u_{i\nu+1} = \Bigg[-(1+m_i)\sum_{\kappa=1}^{\nu} x_{i\kappa}R_{i\nu+1-\kappa} -$$
$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big(\sum_{\kappa=1}^{\nu}(x_{i\kappa}-x_{j\kappa})R_{ij\nu+1-\kappa}+x_{j\kappa}R_{j\nu+1-\kappa}\Big)\Bigg]\Big/\nu,$$

$$V_{i\,\nu+1} = \Big[-(1+m_i)\sum_{\kappa=1}^{\nu} y_{i\kappa} R_{i\,\nu+1-\kappa} -$$

$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big(\sum_{\kappa=1}^{\nu}(y_{i\kappa}-y_{j\kappa})R_{ij\,\nu+1-\kappa} + y_{j\kappa}R_{j\,\nu+1-\kappa}\Big)\Big]/\nu \quad , \qquad (3.102)$$

$$W_{i\,\nu+1} = \Big[-(1+m_i)\sum_{\kappa=1}^{\nu} z_{i\kappa} R_{i\,\nu+1-\kappa} -$$

$$-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big(\sum_{\kappa=1}^{\nu}(z_{i\kappa}-z_{j\kappa})R_{ij\,\nu+1-\kappa} + z_{j\kappa}R_{j\,\nu+1-\kappa}\Big)\Big]/\nu \, ,$$

$$(i=1,\dots n) \quad \nu \geqslant 1 \, .$$

Equations (3.97), at the point $\tau = 0$ give

$$x'_{i\,\tau=0} \equiv u_{i\,\tau=0} = u_{i1} \, ,$$

$$y'_{i\,\tau=0} \equiv V_{i\,\tau=0} = V_{i1} \, ,$$

$$z'_{i\,\tau=0} \equiv W_{i\,\tau=0} = W_{i1} \, ,$$

$$(i=1,\dots n) \, ,$$

and, making use of the equations (3.71), it follows that

$$u_{i1} = x_{i2} \, ,$$

$$V_{i1} = y_{i2} \, , \qquad (3.103)$$

$$W_{i1} = z_{i2} \, ,$$

$$(i=1,\dots n) \, .$$

Summarizing the results of the above calculations the following table is constructed

$$x_{i1} = x_{i\,\tau=0} \, ,$$
$$y_{i1} = y_{i\,\tau=0} \, ,$$
$$z_{i1} = z_{i\,\tau=0} \, ,$$
$$(i=1,\dots n) \, ,$$

$$u_{i1} = x'_{i\ \tau=0},$$
$$v_{i1} = y'_{i\ \tau=0},$$
$$w_{i1} = z'_{i\ \tau=0},$$
$$(i = 1, \dots n),$$
$$r_{i1} = (x_{i1}^2 + y_{i1}^2 + z_{i1}^2)^{1/2},$$
$$r_{ij1} = ((x_{i1} - x_{j1})^2 + (y_{i1} - y_{j1})^2 + (z_{i1} - z_{j1})^2)^{1/2},$$
$$R_{i1} = r_{i1}^{-3},$$
$$R_{ij1} = r_{ij1}^{-3},$$
$$(i, j = 1, \dots n \quad i \neq j),$$

$$x_{i\nu+1} = u_{i\nu}/\nu,$$
$$y_{i\nu+1} = v_{i\nu}/\nu,$$
$$z_{i\nu+1} = w_{i\nu}/\nu,$$
$$(i = 1, \dots n) \quad \nu \geqslant 1,$$

$$u_{i\nu+1} = \Big[-(1+m_i)\sum_{k=1}^{\nu} x_{ik} R_{i\nu+1-k} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \Big(\sum_{k=1}^{\nu}(x_{ik}-x_{jk}) R_{ij\nu+1-k} + x_{jk} R_{j\nu+1-k}\Big)\Big]/\nu,$$

$$v_{i\nu+1} = \Big[-(1+m_i)\sum_{k=1}^{\nu} y_{ik} R_{i\nu+1-k} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \Big(\sum_{k=1}^{\nu}(y_{ik}-y_{jk}) R_{ij\nu+1-k} + y_{jk} R_{j\nu+1-k}\Big)\Big]/\nu,$$

$$w_{i\nu+1} = \Big[-(1+m_i)\sum_{k=1}^{\nu} z_{ik} R_{i\nu+1-k} - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \Big(\sum_{k=1}^{\nu}(z_{ik}-z_{jk}) R_{ij\nu+1-k} + z_{jk} R_{j\nu+1-k}\Big)\Big]/\nu,$$

$$r_{i\nu+1} = \Big[-\sum_{k=2}^{\nu} r_{ik} r_{i\nu+2-k}(\nu+1-k) + \sum_{k=1}^{\nu}(\nu+1-k)(x_{ik}x_{i\nu+2-k} + y_{ik}y_{i\nu+2-k} + z_{ik}z_{i\nu+2-k})\Big]/\nu r_{i1},$$

$$r_{ij\,\nu+1} = \left[-\sum_{\kappa=2}^{\nu} r_{ij\kappa} r_{ij\,\nu+2-\kappa}(\nu+1-\kappa) + \sum_{\kappa=1}^{\nu}(\nu+1-\kappa)\Big((x_{i\kappa}-x_{j\kappa})(x_{i\,\nu+2-\kappa}-x_{j\,\nu+2-\kappa}) + (y_{i\kappa}-y_{j\kappa})(y_{i\,\nu+2-\kappa}-y_{j\,\nu+2-\kappa}) + (z_{i\kappa}-z_{j\kappa})(z_{i\,\nu+2-\kappa}-z_{j\,\nu+2-\kappa})\Big)\right] \Big/ \nu r_{ij1} \,,$$

$$R_{i\,\nu+1} = \left[-\sum_{\kappa=2}^{\nu}(\nu+1-\kappa) r_{i\kappa} R_{i\,\nu+2-\kappa} - 3\sum_{\kappa=1}^{\nu} R_{i\kappa} r_{i\,\nu+2-\kappa}(\nu+1-\kappa)\right] \Big/ \nu r_{i1} \,,$$

$$R_{ij\,\nu+1} = \left[-\sum_{\kappa=2}^{\nu}(\nu+1-\kappa) r_{ij\kappa} R_{ij\,\nu+2-\kappa} - 3\sum_{\kappa=1}^{\nu}(\nu+1-\kappa) R_{ij\kappa} r_{ij\,\nu+2-\kappa}\right] \Big/ \nu r_{ij1} \,,$$

$$(i, j = 1, \ldots n \quad i \neq j) \quad \nu \geqslant 1 .$$

If the equations are used in the order where they are given above, they are perfectly recurrent, that is, each coefficient depends only on the preceding coefficients. All the first-order coefficients are given by the initial conditions. After the coefficients of the series (3.87) and (3.92) are found we substitute these series in the equations (3.49) to obtain $A_{i\kappa}(\tau)$ $(i, \kappa = 1, \ldots n)$ as power series, the introduction of which in the equations (3.53), (3.54) enables us to solve the system (3.53), (3.54) (by equating the coefficients of terms with the same power of τ. Since f_{ij}, g_{ij} $(i, j = 1, \ldots n)$ are known, we proceed then to the second approximation. This process is repeated until successive approximations of the velocities agree up to the desired accuracy.

In the special case of the two-body problem the equation of motion is

$$\underset{\sim}{r}'' = -\mu \frac{\underset{\sim}{r}}{r^3} . \tag{3.104}$$

Let us consider the wellknown equation

$$\underset{\sim}{r} = f \underset{\sim}{r}_0 + g \underset{\sim}{r}_0' \,, \tag{3.105}$$

where f and g are series in powers of the independent variable τ, and $(-\tau^{**}, \tau^{**})$ is their common interval of convergence.

Since $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_0'$ are constant vectors, we can differentiate equation (3.105) once with respect to τ to obtain the velocity vector

$$\underset{\sim}{r}' = f'\underset{\sim}{r}_0 + g'\underset{\sim}{r}_0' \qquad \tau \in (-\tau^{**}, \tau^{**}) \, . \tag{3.106}$$

Differentiating once more gives

$$\underset{\sim}{r}'' = f''\underset{\sim}{r}_0 + g''\underset{\sim}{r}_0' \qquad \tau \in (-\tau^{**}, \tau^{**}) \, , \tag{3.107}$$

which upon substitution of equation (3.104) becomes

$$-\mu \frac{\underset{\sim}{r}}{r^3} = f''\underset{\sim}{r}_0 + g''\underset{\sim}{r}_0' \qquad \tau \in (-\tau^{**}, \tau^{**}) \, ,$$

or

$$\underset{\sim}{r} = -\frac{r^3}{\mu} f''\underset{\sim}{r}_0 - \frac{r^3}{\mu} g''\underset{\sim}{r}_0' \qquad \tau \in (-\tau^{**}, \tau^{**}) \, . \tag{3.108}$$

Subtraction of equation (3.108) from equation (3.105) gives

$$\left(f + \frac{r^3}{\mu} f''\right)\underset{\sim}{r}_0 + \left(g + \frac{r^3}{\mu} g''\right)\underset{\sim}{r}_0' = 0 \qquad \tau \in (-\tau^{**}, \tau^{**}) \, . \tag{3.109}$$

Since the vectors $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_0'$ are independent, their coefficients in the relation (3.109) must be equal to zero. Therefore,

$$\begin{aligned} f &= -\frac{r^3}{\mu} f'' \, , \\ g &= -\frac{r^3}{\mu} g'' \, , \qquad \tau \in (-\tau^{**}, \tau^{**}) \, . \end{aligned} \tag{3.110}$$

Suppose that f and g have the form

$$f=\sum_{\nu=0}^{\infty} f_\nu \frac{\tau^\nu}{\nu!},$$
$$g=\sum_{\nu=0}^{\infty} g_\nu \frac{\tau^\nu}{\nu!}, \tag{3.111}$$
$$\tau\in(-\tau^{**},\tau^{**}),$$

then differentiating these equations twice with respect to τ gives

$$f''=\sum_{\nu=2}^{\infty}(\nu-1)\,\nu\, f_\nu \frac{\tau^{\nu-2}}{\nu!},$$

$$g''=\sum_{\nu=2}^{\infty}(\nu-1)\,\nu\, g_\nu \frac{\tau^{\nu-2}}{\nu!},$$

or

$$f''=\sum_{\nu=2}^{\infty} f_\nu \frac{\tau^{\nu-2}}{(\nu-2)!},$$

$$g''=\sum_{\nu=2}^{\infty} g_\nu \frac{\tau^{\nu-2}}{(\nu-2)!},$$

or

$$f''=\sum_{\nu=0}^{\infty} f_{\nu+2} \frac{\tau^{\nu}}{\nu!},$$

$$g''=\sum_{\nu=0}^{\infty} g_{\nu+2} \frac{\tau^{\nu}}{\nu!}, \tag{3.112}$$
$$\tau\in(-\tau^{**},\tau^{**}).$$

Equations (3.110), upon substitution of equations (3.112) become

$$\sum_{\nu=0}^{\infty} f_\nu \frac{\tau^\nu}{\nu!}=-\frac{r^3}{\mu}\sum_{\nu=0}^{\infty} f_{\nu+2}\frac{\tau^\nu}{\nu!},$$
$$\tau\in(-\tau^{**},\tau^{**})$$
$$\sum_{\nu=0}^{\infty} g_\nu \frac{\tau^\nu}{\nu!}=-\frac{r^3}{\mu}\sum_{\nu=0}^{\infty} g_{\nu+2}\frac{\tau^\nu}{\nu!}.$$

Consider r^3 as constant in the interval $(-\tau^{**},\tau^{**})$ (an accurate approximation for small intervals of time), the coefficients of terms of the same power of τ must be equal. That is

$$f_\nu=-\frac{r^3}{\mu}f_{\nu+2},$$
$$\nu\geqslant 0$$
$$g_\nu=-\frac{r^3}{\mu}g_{\nu+2}. \tag{3.113}$$

Analytically, for $\nu = 0$, $\nu = 1$, $\nu = 2$, ... we find the following

$$
\begin{array}{llll}
f_2 = -\frac{\mu}{r^3} f_0 & f_3 = -\frac{\mu}{r^3} f_1 & g_2 = -\frac{\mu}{r^3} g_0 & g_3 = -\frac{\mu}{r^3} g_1 \\
f_4 = -\frac{\mu}{r^3} f_2 & f_5 = -\frac{\mu}{r^3} f_3 & g_4 = -\frac{\mu}{r^3} g_2 & g_5 = -\frac{\mu}{r^3} g_3 \\
\vdots & \vdots & \vdots & \vdots \\
f_{2\lambda} = -\frac{\mu}{r^3} f_{2\lambda-2} & f_{2\lambda+1} = -\frac{\mu}{r^3} f_{2\lambda-1} & g_{2\lambda} = -\frac{\mu}{r^3} g_{2\lambda-2} & g_{2\lambda+1} = -\frac{\mu}{r^3} g_{2\lambda-1} \\
\vdots & \vdots & \vdots & \vdots
\end{array} \tag{3.114}
$$

Obviously, the knowledge of the coefficients of the first two terms, namely f_0 , g_0 , f_1 , g_1 , permits all the coefficients of the series (3.111) to be calculated.

The coefficients f_0 and g_0 are obtained as follows: At time $\tau = 0$ equations (3.111) and (3.105) give

$$
\begin{aligned}
f_{\tau=0} &= f_0 \;, \\
g_{\tau=0} &= g_0 \;, \\
\underset{\sim}{r}_{\tau=0} &= f_{\tau=0}\, \underset{\sim}{r}_0 + g_{\tau=0}\, \underset{\sim}{r}_0' \;,
\end{aligned}
$$

or

$$
\underset{\sim}{r}_0 = f_0 \underset{\sim}{r}_0 + g_0 \underset{\sim}{r}_0' \;,
$$

which yields

$$f_0 = 1 \quad , \quad g_0 = 0 \quad , \tag{3.115}$$

since $\underset{\sim}{\Gamma_0} \equiv \underset{\sim}{\Gamma_{\tau=0}}$.

On the other hand, equation (3.106), at time $\tau = 0$ gives

$$\underset{\sim}{\Gamma'_{\tau=0}} = f'_{\tau=0}\underset{\sim}{\Gamma_0} + g'_{\tau=0}\underset{\sim}{\Gamma'_0} \, . \tag{3.116}$$

Since $\underset{\sim}{\Gamma'_{\tau=0}} \equiv \underset{\sim}{\Gamma'_0}$ and because $f'_{\tau=0} = f_1$ and $g'_{\tau=0} = g_1$ equation (3.116) becomes

$$\underset{\sim}{\Gamma'_0} = f_1\underset{\sim}{\Gamma_0} + g_1\underset{\sim}{\Gamma'_0} \, ,$$

which yields

$$f_1 = 0 \quad , \quad g_1 = 1 \, . \tag{3.117}$$

Introduction of these values in the equations (3.114) gives

$$f_{2\lambda} = \left(-\frac{\mu}{r^3}\right)^{\lambda} , \; f_{2\lambda+1} = 0 \; , \; g_{2\lambda} = 0 \; , \; g_{2\lambda+1} = \left(-\frac{\mu}{r^3}\right)^{\lambda} \quad \lambda \geqslant 0 \, , \tag{3.118}$$

and hence the f and g series have the form

$$\left.\begin{aligned} f &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \left(\frac{\mu}{r^3}\right)^{\nu} \frac{\tau^{2\nu}}{(2\nu)!} \, , \\ g &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \left(\frac{\mu}{r^3}\right)^{\nu} \frac{\tau^{2\nu+1}}{(2\nu+1)!} \, , \end{aligned}\right\} \quad \tau \in (-\tau^{**}, \tau^{**}) \tag{3.119}$$

or

$$f = \sum_{\nu=0}^{\infty} (-1)^{\nu} \left(\sqrt{\frac{\mu}{r^3}}\, \tau\right)^{2\nu} \Big/ (2\nu)! \, , \tag{3.120}$$

$$g=\left\{\sum_{\nu=0}^{\infty}(-1)^{\nu}\left(\sqrt{\frac{\mu}{r^3}}\tau\right)^{2\nu+1}\Big/(2\nu+1)!\right\}\sqrt{\frac{r^3}{\mu}}\ ,\qquad \tau\in(-\tau^{**},\tau^{**})\ .$$

If we now apply the series for $\sin x$ and $\cos x$

$$\cos x=\sum_{\nu=0}^{\infty}(-1)^{\nu}x^{2\nu}/(2\nu)!\ ,$$

$$\sin x=\sum_{\nu=0}^{\infty}(-1)^{\nu}x^{2\nu+1}/(2\nu+1)!\ ,$$

then equations (3.120) give

$$f=\cos\left(\sqrt{\frac{\mu}{r^3}}\tau\right)\ ,$$

$$g=\sqrt{\frac{r^3}{\mu}}\sin\left(\sqrt{\frac{\mu}{r^3}}\tau\right)\ , \qquad \tau\in(-\tau^{**},\tau^{**})\ . \tag{3.121}$$

Consequently, two known position vectors and their corresponding times, say $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$, t_1 , t_2 , permits the calculation of the power series (3.121) (assuming $\underset{\sim}{r}$ = constant $= r_1$ or r_2). Solving equation (3.105) with respect to $\underset{\sim}{r}_0'$ gives the velocity of the body.

Let us return now to the n-body problem. Clearly, knowledge of the position vectors of the bodies at two different instants enables us to calculate the initial conditions of the problem and to form the series (3.68), (3.97). By inserting in these series a value, say $\Delta\tau$, of the variable τ , we can calculate the position and velocity vectors of the bodies at a subsequent instant, $\Delta\tau/k$ days later. Taking these new values as initial conditions we repeat the process, thereby detailing the paths of the bodies in space.

Chapter 4

Reduction of the plates and numerical results

The first stage in the numerical application of the above, briefly described methods, is the measurement of the photographic plates and their reduction.

For such a purpose the first task to be undertaken was the identification of the minor planet on the plates. This was accomplished using a Blink Comparator. Two plates, with the same guiding star, were used each time.

After that, the measurement of the plates was carried out on a Zeiss measuring machine. All the stars on the plates for which the right ascension and declination are given in the Star Catalogue of the Smithsonian Astrophysical Observatory were taken as reference stars.

We give, in brief, the method of reduction of the measurements and the applied corrections.

A photographic plate (free from errors) is the central projection of a part of the celestial sphere from its centre (ex. the centre of the object glass) on to the tangent plane at a point defined by the intersection of the optical axis of the instrument with the celestial sphere. The rectangular co-ordinates in this plane, called the "Standard co-ordinates", can be defined as follows:

(a) The co-ordinate system lies in a plane tangent to the celestial sphere.

(b) The origin of the system is the tangent point T .

(c) The Y-axis is tangent to the declination circle through T. Its positive direction is in the direction in which the distance of T from the north celestial sphere is less than 180°.

(d) The X-axis is perpendicular to the Y-axis at T. Its positive direction is eastward of the declination circle, hence the increasing value of X corresponds to increasing right ascension.

(e) The unit of length is the radius of the sphere (in practice the focal length, f).

(f) The standard co-ordinates X and Y of a point S on the celestial sphere are the rectangular co-ordinates, in the defined system, of the central projection of S on the tangent plane from the centre of the sphere.

The co-ordinates of the stars that are obtained from the measurement of the plates are usually called the "measured co-ordinates".

The aim of the reduction is the derivation of the spherical co-ordinates α, δ of the objects of interest on the plate. This is possible because there is one-to-one correspondence between the points on the sphere and the points on the tangent plane. As a rule the reduction is obtained in two steps. Firstly, we transform the measured co-ordinates into standard co-ordinates, then these are transformed into spherical co-ordinates α, δ

In practice, we make the measured co-ordinates a close approximation to the standard. Then only small corrections are needed. These corrections may be divided into groups: "instrumental" corrections, depending on the state of the telescope while taking the plates, namely, scale, orientation, zero point and plate skewness; and "spherical corrections", due to astronomical causes, namely, refraction, aberration, precession and nutation.

For the second group of corrections formulae can be derived from which they can be computed with sufficient accuracy, on the basis of the known exposure data. For the first group, however, at least some of the coefficients have to be determined empirically, using the known positions of the reference stars.

The displacement of the image of a star on the plate from the position corresponding to its standard co-ordinates is given, generally with sufficient accuracy, as a linear expression in the co-ordinates. Hence, if X and Y are the standard co-ordinates of a star, and x, y its measured co-ordinates, we can write

$$\Delta x \equiv X - x = A x + B y + C ,$$
$$\Delta y \equiv Y - y = A^* x + B^* y + C^* . \tag{4.1}$$

It is possible to go a step further and to introduce quadratic terms, so the most general equations then take the form

$$\Delta x = A x + B y + C + D x^2 + E x y + F y^2 ,$$
$$\Delta y = A^* x + B^* y + C^* + D^* x^2 + E^* x y + F^* y^2 . \tag{4.2}$$

The introduction of these additional constants is not without its disadvantages, since a correspondingly larger number of reference stars will be required to determine them with adequate precision.

In order to achieve the second step of the reduction, that is, the transformation of the standard co-ordinates into spherical co-ordinates, α, δ we make use of the formulae

$$\tan(\alpha - \alpha_0) = \frac{X}{\cos\delta_0 - Y\sin\delta_0} ,$$
$$\sin\delta = \frac{\sin\delta_0 + Y\cos\delta_0}{(1 + X^2 + Y^2)^{1/2}} , \tag{4.3}$$

where α_0 and δ_0 are the right ascension and declination of the origin of the standard co-ordinates, and α, δ the right ascension and declination of the star with standard co-ordinates X and Y, all of which are referred to the mean equinox of 1950.0.

The formulae, which give the standard co-ordinates X, Y as functions of the spherical equatorial co-ordinates, have the form

$$X = \frac{\cos\delta \sin(\alpha - \alpha_0)}{\sin\delta_0 \sin\delta + \cos\delta_0 \cos\delta \cos(\alpha - \alpha_0)} ,$$

$$Y = \frac{\sin\delta \cos\delta_0 - \sin\delta_0 \cos\delta \cos(\alpha - \alpha_0)}{\sin\delta \sin\delta + \cos\delta_0 \cos\delta \cos(\alpha - \alpha_0)} . \tag{4.4}$$

The calculations were carried out using the IBM 360/44 computer of St Andrews University and the IBM 370/165computer of Cambridge University (for the power series method). When writing the programs for the computations, a deliberate attempt was made to significantly reduce the time required for the calculations.

The numerical results are tabulated as follows:

The first five tables 1a, 1b, 1c, 1d, 1e refer to five different photographic plates, taken on different dates. These tables include the following information for each reference star:

(a) A serial number (A/A).

(b) The numbers which the stars possess in the Star Catalogue of the Smithsonian Astrophysical Observatory (SAOSC).

(c) The visual magnitude of the star (MAGN).

(d) The right ascension (RA) in hours, minutes and seconds, and the declination (DEC) in degrees, minutes and seconds of arc, for equator equinox and epoch 1950.0.

(e) The annual proper motion for right ascension (MA) and declination (MD) in seconds of time and seconds of arc respectively.

(f) The date of observation (DATE) and the interval of time, in tropical year, elapsed since 1950.0 (T).

The first two lines give the values for the guiding star.

The tables 2a, 2b, 2c, 2d, 2e, give:

(a) The serial number for each reference star (A/A).

(b) The numbers which the stars possess in the Star Catalogue of the Smithsonian Astrophysical Observatory (SAOSC).

(c) The visual magnitude of the star (MAGN).

(d) The right ascension (RA) in hours, minutes and seconds, and the declination (DEC) in degrees, minutes and seconds of arc, for the date of observation and the equator and equinox of epoch 1950.0.

(e) The standard co-ordinates of the stars (X, Y) in seconds of time and seconds of arc respectively and the same quantities in mm.

(f) The date of observation (DATE).

The first line gives the values for the guiding star.

In order to find the standard co-ordinates of the stars the formulae (4.4) were used.

The tables 3a, 3b, 3c, 3d, 3e, give

(a) The serial number (A/A).

(b) The numbers which the stars possess in the Star Catalogue of the Smithsonian Astrophysical Observatory (SAOSC).

(c) The measured co-ordinates of the stars (X-XO, Y-YO) with respect to the guiding star and the values taken from the measurement (X, Y), in mm.

The first line gives the date of observation (DATE) and the number which is given in the guiding star in the Star Catalogue of the Smithsonian Astrophysical Observatory (SAOSC(G.S.)).

The first two lines of the columns refer to the guiding star and the minor planet respectively.

The value of each co-ordinate is the mean value of at least ten measurements. To obtain great accuracy, we measured each co-ordinate in two positions of the plate, differing by 180°.

In table 4 the following information is given:

(a) The serial number (A/A).

(b) The numbers which the stars possess in the Star Catalogue of the Smithsonian Astrophysical Observatory (SAOSC).

(c) The right ascension (RA) in hours, minutes and seconds and the declination (DEC) in degrees, minutes and seconds of arc, as calculated from the measured co-ordinates.

(d) The difference of the calculated right ascension and declination from the corresponding values given in the tables 2a, 2b, 2c, 2d, 2e (DRA, DDEC), in seconds of time and seconds of arc respectively.

In order to calculate the right ascension and the declination of the stars from their measured co-ordinates, we transformed the measured co-ordinates into standard co-ordinates using the corrections given by relations (4.1). The least squares method was used for the calculation of the coefficient of formulae (4.1). Then the relations (4.3) give the spherical co-ordinates

Values for the right ascension and the declination have been derived by using the method of Dependences. These values are of slightly less accuracy than the values derived by using the least squares method.

Table 4 also gives results using stars distributed symmetrically around the guiding star.

Three stars were used as test stars for each photographic plate.

The results in the table 4 are given in five different groups, which correspond to the five different photographic plates.

Table 5 gives the following information for each of the observations:

(a) The day of observation (DA).

(b) The local mean sidereal time (MTOO(LMST)) and the Greenwich mean sidereal time (MTOO (GMST)), which corresponds to the mid-point of the exposure, in hours, minutes and seconds.

(c) The Greenwich mean sidereal time at 0^h U.T. (on the date of observation), in hours, minutes and seconds (GMST AT O U.T.).

(d) The difference between the Greenwich mean sidereal time, corresponding to the mid-point of the observation, and the Greenwich mean sidereal time at 0^h U.T., in hours, minutes and seconds. This difference is given in mean sidereal time (DIFFER.(MST)) and in mean solar time (DIFFER. (MSL.T.)).

(e) The month of observation (MH).

(f) The Universal time (UT), the correction $\Delta T = E.T. - U.T.$ (in second of time), and the ephemeris time (ET).

(g) The U.T. and the E.T. measured from the first day of the month of the first observation (UT*, ET*).

(h) The values of T_0 and T (TO, T), in tropical countries, where T_0 is the time elapsed since 1900.0 and T the time elapsed from the initial epoch to the epoch 1950.0.

(i) The angles ζ_0 and z (ZO, Z) in seconds of time.

(j) The right ascension (RA) of the minor planet, in hours, minutes and seconds and the right ascension (RA*) and the declination (DEC) of the minor planet, in degrees, minutes and seconds of arc.

Tables 6a, 6b, 6c, 6d, 6e, give the topocentric rectangular co-ordinates of the Sun on the date of observation (X, Y, Z) and the components (λ, μ, ν) of the unit-vector $\hat{\underset{\sim}{\rho}}$ along the topocentric position vector of the planet (LAMDA, MI, NI).

To obtain the topocentric rectangular co-ordinates of the Sun on the date of observation we made use of the Everett's interpolation formula

$$f_p = (1-p)\, f_0 + p\, f_1 + E_2 \delta_0^2 + F_2 \delta_1^2 ,$$

and introduced the topocentric corrections (DX, DY, DZ), given by the relations

$$\Delta X = \Delta_{xy} \cos\theta , \quad \Delta Y = \Delta_{xy} \sin\theta , \quad \Delta Z = \Delta_z .$$

The coefficients E_2 and F_2 are given by the formulae

$$E_2 = (1-P)\,\frac{(1-P)^2-1}{6}\,, \quad F_2 = P\,\frac{P^2-1}{6}\,,$$

and the factors Δxy and Δz are given, for each observatory, in the Astronomical Ephemeris.

θ is the local sidereal time, with respect to the mean equinox 1950.0, hence

$$\theta = L.S.T. + M\,,$$

where L.S.T. is the local sidereal time, with respect to the equinox of the date, and $M = \zeta_0 + z$ the general precession in right ascension.

Tables 7a, 7b, 7c, 7d refer to the four approximations of the Gaussian method. These tables give

(a) The modified times τ_1, τ_2, τ_3, (T1, T2, T3).

(b) The values of the quantities B_1, B_2, B_3, $1+B_1 r_1^{-3}$, $1-B_2 r_2^{-3}$, $1+B_3 r_3^{-3}$, C_1, C_3, V_1 and V_3.

(c) The solution of the system of the equations (1.8) and (1.21) by iteration (P2, R2).

(d) The distances $\rho_1, \rho_2, \rho_3, R_1, R_2, R_3$, (P1, P2, P3, R1, R2, R3) and the components of the vectors $\underset{\sim}{R}_1, \underset{\sim}{R}_2, \underset{\sim}{R}_3$.

(e) The final values of the constants C_1, C_3. Generally, the final values are determined when successive approximations differ by less than 10^{-10}.

Table 8 gives

(a) The values of the quantities k_i, m_i, l_i $(i = 1, 2, 3)$, (K1, K2, K3, M1, M2, M3, L1, L2, L3).

(b) The first approximation for the quantities $h_i, \tilde{y}_i$ $(i = 1, 2, 3)$, (H1, H2, H3, Y1, Y2, Y3).

(c) The second approximation of the same quantities, including the values of x_i, ξ_i $(i=1,2,3)$, (X1,X2,X3,KS1,KS2,KS3).

(d) The values of C_1 and C_3, obtained from the relations,

$$C_1 = \frac{\tau_1 \tilde{y}_2}{\tau_2 \tilde{y}_1}, \quad C_3 \quad \frac{\tau_3 \tilde{y}_2}{\tau_2 \tilde{y}_3}.$$

These differ by less than 10^{-9} from the values of the last approximation (Table 7d)

Tables 9a and 9b give the final results of the Gauss method. Table 9b is obtained as follows: After the values of $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ are known we take instead of relations (1.19) the relations

$$\nu_1 = C_1 r_2^{\,3}\left(\frac{\tilde{y}_2}{\tilde{y}_1} - 1\right), \quad \nu_3 = C_3 r_2^{\,3}\left(\frac{\tilde{y}_2}{\tilde{y}_3} - 1\right),$$

and we proceed as described in the first chapter.

Tables 10a and 10b refer to Laplace's method. These tables give the coefficients and the solutions of systems (2.18) (2.19) and (2.20) before and after the correction for the planetary aberration. Notice that the data from all the observations are used to find the values of the coefficients given in tables 10a, 10b.

Tables 11a and 11b give

(a) The values of the determinants D, D_1, D_2, (D, D1, D2).

(b) The solution of the system of equations (2.23) and (2.24) by iteration (RO, PO).

(c) The modified times τ_1, τ_2, (T1, T2).

(d) The values of ρ_0, R_0 and ρ_0', (PO, RO, POD).

Table 11b gives the above quantities after the correction for the planetary aberration. It also gives the components (VX, VY, VZ) of the velocity vector.

Tables 12a and 12b give the final results of Laplace's method. We obtained Table 12b by using relations (3.121) and (3.105) to find the velocity vector

Table 13 gives the heliocentric rectangular equatorial co-ordinates (for the mean equinox and equator 1950.0) of the planets Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and the minor planet, for the dates 1971 Nov 14, 20, 25 at 0^h E.T.

In order to find these values the apparent right ascension and the apparent declination, given in the Astronomical Ephemeris, were transformed into mean right ascension and declination by using first and second order corrections. Then, using the true distances of planets from the Earth and the heliocentric equatorial rectangular co-ordinates of the Earth for mean equinox and equator 1950.0 and for the dates 1971 Nov 14, 20, 21 at 0^h E.T., the values given in the table are obtained.

The co-ordinates of the minor planet were found, by correcting the values given in Table 7d.

Table 14 gives

(a) The velocities of the bodies.

(b) The total initial energy of the system.

(c) The heliocentric equatorial rectangular co-ordinates of the bodies for equinox and equator 1950.0, and for 0^h and 12^h E.T.

(d) The total energy for every step (C).

(e) The right ascension and declination of the minor planet (RA, DEC) for mean equinox and equator 1950.0.

The ephemeris of the asteroid were constructed by solving the equations

$$\xi = \rho \cos\delta \cos\alpha \, , \quad \eta = \rho \cos\delta \sin\alpha \, , \quad \zeta = \rho \sin\delta ,$$

where $\underset{\sim}{\rho} = (\xi, \eta, \zeta)$ is the geocentric position vector of the asteroid, obtained from the heliocentric position vectors of the Earth and the Asteroid.

TABLE 1a

GUIDING STAR SAOSC 110723 MAGN 4.4 RA 2^h 42^m $14^s.102$ DEC $9°$ $54'$ $15''.02$ DATE 1971.8707670

MA= $0^s.0190$ MD= $-0''.030$ T= 21.8707670 T.Y.

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC '	DEC "	MA s	MD "
1	93088	8.7	2	42	50.547	10	35	56.21	0.0008	-0.012
2	93091	8.8	2	43	13.671	10	10	46.77	0.0060	-0.019
3	93110	8.5	2	44	55.640	10	1	6.43	0.0033	-0.053
4	93136	8.8	2	47	43.627	10	25	24.69	0.0114	-0.012
5	93139	8.9	2	47	52.315	11	12	1.96	-0.0025	-0.021
6	93150	8.5	2	49	15.023	11	6	8.62	0.0009	-0.012
7	93151	8.6	2	49	15.439	11	10	17.46	0.0020	-0.422
8	93152	8.7	2	49	23.043	10	35	28.04	-0.0016	-0.056
9	93158	9.0	2	49	52.533	10	45	28.00	0.0005	0.005
10	93161	8.8	2	50	10.111	10	27	56.42	0.0028	-0.015
11	93162	8.7	2	50	10.519	10	27	54.17	0.0011	0.003
12	110737	8.9	2	43	43.725	9	11	29.87	0.0012	0.014
13	110742	9.1	2	44	11.597	8	57	58.93	0.0085	0.059
14	110743	9.0	2	44	18.686	8	32	28.09	0.0006	0.005
15	110745	8.0	2	44	42.089	9	6	0.27	0.0006	-0.043
16	110752	8.7	2	45	12.423	8	43	0.72	0.0071	0.005
17	110753	8.7	2	45	18.273	8	41	27.63	0.0012	-0.007
18	110758	9.0	2	45	28.476	8	41	40.46	0.0024	0.012
19	110662	9.5	2	36	40.478	9	16	26.84	0.0051	0.032
20	110674	9.0	2	37	55.511	9	17	25.74	0.0042	-0.006
21	110693	8.8	2	39	52.986	9	35	27.79	0.0010	0.023
22	110702	9.7	2	40	39.224	9	20	12.94	0.0066	-0.034
23	110716	8.7	2	41	44.984	9	1	10.21	-0.0001	-0.013
24	93034	6.9	2	36	21.619	10	25	22.78	0.0029	-0.058
25	93036	9.0	2	36	29.123	10	27	25.04	0.0007	0.0
26	93050	8.7	2	38	5.888	10	8	13.36	0.0	0.003
27	93059	6.7	2	39	18.507	10	19	56.62	-0.0014	-0.025
28	93067	6.3	2	39	47.143	10	31	46.87	-0.0018	-0.018
29	93071	8.7	2	40	24.728	10	28	45.13	0.0012	-0.027
30	93078	8.9	2	41	14.472	10	2	46.59	0.0001	0.037

TABLE 1b

GUIDING STAR SAOSC 110698 MAGN 8.8 RA $2^h\ 39^m\ 52^s.986$ DEC $9^\circ\ 35'\ 27''.79$ DATE 1971.3870742

MA= $0^s.0010$ MD= $0''.023$ T= 21.8870742 T.Y.

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	MA s	MD ″
1	110723	4.4	2	42	14.102	9	54	15.02	0.0190	-0.030
2	93071	8.7	2	40	24.728	10	28	45.13	0.0012	-0.027
3	93078	8.9	2	41	14.472	10	2	46.59	0.0001	0.037
4	93088	8.7	2	42	50.547	10	35	56.21	0.0008	-0.012
5	93091	8.8	2	43	13.671	10	10	46.77	0.0060	-0.019
6	93110	8.5	2	44	55.640	10	1	6.43	0.0033	-0.053
7	110702	9.7	2	40	39.224	9	20	12.94	0.0066	-0.034
8	110716	8.7	2	41	44.984	9	1	10.21	-0.0001	-0.013
9	110737	8.9	2	43	43.725	9	11	29.87	0.0012	0.014
10	110742	9.1	2	44	11.597	8	57	58.93	0.0035	0.059
11	110743	9.0	2	44	18.686	8	32	28.09	0.0006	0.005
12	110745	8.0	2	44	42.089	9	6	0.27	0.0006	-0.043
13	110752	8.7	2	45	12.423	8	43	0.72	0.0071	0.005
14	110753	8.7	2	45	18.273	8	41	27.63	0.0012	-0.007
15	110646	8.6	2	34	44.670	9	20	44.08	-0.0012	0.010
16	110650	8.7	2	35	5.804	8	44	41.95	0.0012	0.008
17	110658	8.3	2	36	19.201	8	42	17.78	0.0005	-0.018
18	110662	9.5	2	36	40.478	9	16	26.84	0.0051	0.032
19	110674	9.0	2	37	55.511	9	17	25.74	0.0042	-0.006
20	110686	7.8	2	38	43.622	8	18	52.10	-0.0007	-0.032
21	110632	8.7	2	33	0.996	9	54	18.62	0.0020	-0.011
22	93026	8.8	2	34	7.871	10	19	19.14	0.0013	-0.004
23	93034	6.9	2	36	21.619	10	25	22.74	0.0029	-0.058
24	93036	9.0	2	36	29.123	10	27	25.04	0.0007	0.0
25	93050	8.7	2	38	5.888	10	8	13.36	0.0	0.003
26	93059	6.7	2	39	18.507	10	19	56.62	-0.0014	-0.025
27	93067	6.3	2	39	47.143	10	31	46.85	-0.0018	-0.018

TABLE 1c

GUIDING STAR	SAOSC	MAGN	RA			DEC			DATE
	110650	8.7	2^h	35^m	$5^s.804$	8°	$44'$	$41''.95$	1971.9009253

$MA = 0^s.0012$ $MD = 0''.008$ $T = 21.9009253$ T.Y.

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	MA s	MD ″
1	110662	9.5	2	36	40.478	9	16	26.84	0.0051	0.032
2	110674	9.0	2	37	55.511	9	17	25.74	0.0042	-0.006
3	110698	8.8	2	39	52.986	9	35	27.79	0.0010	0.023
4	110702	9.7	2	40	39.224	9	20	12.94	0.0066	-0.034
5	110716	8.7	2	41	44.984	9	1	10.21	-0.0001	-0.013
6	110653	6.5	2	35	21.099	7	28	47.80	0.0056	-0.039
7	110658	8.3	2	36	19.201	8	42	17.74	0.0005	-0.018
8	110663	8.5	2	36	43.656	8	7	49.67	0.0049	-0.005
9	110667	8.4	2	37	6.006	7	59	19.79	-0.0005	-0.013
10	110670	9.2	2	37	22.388	7	33	4.86	-0.0017	0.025
11	110678	8.5	2	38	14.720	8	1	35.72	-0.0013	0.0
12	110686	7.8	2	38	43.622	8	18	52.10	-0.0007	-0.032
13	110696	8.9	2	39	46.561	7	59	42.93	-0.0007	0.006
14	110606	8.6	2	30	19.543	7	40	6.61	-0.0005	0.022
15	110609	8.7	2	30	35.620	8	9	30.26	0.0022	-0.031
16	110610	9.0	2	30	39.993	8	26	9.52	-0.0015	-0.012
17	110640	6.0	2	33	55.925	7	30	47.39	-0.0035	-0.027
18	110643	9.5	2	34	30.283	8	16	0.97	-0.0005	-0.007
19	110647	8.6	2	34	52.854	7	23	16.02	0.0019	-0.002
20	110648	8.6	2	34	55.364	7	20	14.07	0.0008	0.015
21	110601	8.7	2	30	3.794	9	37	34.20	0.0016	-0.002
22	110615	7.7	2	31	32.786	8	46	11.70	0.0020	0.022
23	110632	8.7	2	33	0.996	9	54	18.62	0.0020	-0.011
24	110639	8.7	2	33	51.614	8	55	9.16	0.0051	-0.006
25	110646	8.6	2	34	44.670	9	20	44.08	-0.0012	0.010

TABLE 1d

GUIDING STAR	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	DATE
	93186	8.2	2	52	41.149	10	15	21.02	1971.8240538

$MA = -0\overset{s}{.}0012$ $MD = -0\overset{''}{.}008$ T= 21.8240538 T.Y.

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	MA s	MD ″
1	93198	8.7	2	54	14.403	10	26	40.77	-0.0001	0.011
2	93200	8.7	2	54	25.131	10	36	12.22	-0.0009	0.013
3	93204	8.6	2	54	39.749	10	16	52.66	-0.0009	-0.058
4	93227	8.5	2	57	7.782	10	13	17.95	0.0034	-0.039
5	93228	8.0	2	57	22.177	10	27	32.93	0.0091	-0.072
6	93231	8.9	2	57	52.298	10	36	7.84	-0.0024	0.0
7	93232	6.2	2	58	1.160	10	40	24.51	0.0053	-0.028
8	93244	8.5	2	59	5.790	10	39	20.56	0.0028	-0.046
9	93212	7.8	2	55	41.918	10	0	20.24	0.0015	0.013
10	93223	8.3	2	56	35.508	10	0	42.64	-0.0004	-0.004
11	93234	8.1	2	58	6.877	10	2	49.35	0.0003	-0.021
12	110840	8.4	2	52	51.084	9	36	47.07	0.0001	-0.054
13	110859	8.3	2	53	47.664	9	55	30.29	0.0017	-0.001
14	110862	8.0	2	54	18.827	9	57	44.69	-0.0046	-0.083
15	110888	8.6	2	56	58.919	9	26	13.23	0.0018	-0.049
16	110783	9.0	2	48	37.656	9	11	18.96	0.0003	-0.001
17	110809	8.7	2	50	12.765	8	57	20.23	-0.0015	-0.003
18	110817	6.8	2	51	4.867	9	7	58.08	0.0038	-0.023
19	110824	8.6	2	51	43.469	9	34	13.19	0.0009	0.025
20	110834	8.8	2	52	33.359	8	54	35.70	0.0021	-0.017
21	93134	8.6	2	47	35.411	11	24	32.33	-0.0004	-0.006
22	93136	8.3	2	47	43.627	10	25	24.69	0.0114	-0.012
23	93139	8.9	2	47	52.315	11	12	1.96	-0.0025	-0.021
24	93150	8.5	2	49	15.023	11	6	8.62	0.0009	-0.012
25	93151	8.6	2	49	15.439	11	10	17.46	0.0020	-0.442
26	93152	8.7	2	49	23.043	10	35	28.04	-0.0016	-0.056
27	93158	9.0	2	49	52.533	10	45	28.00	0.0005	0.005
28	93161	8.8	2	50	10.111	10	27	56.42	0.0028	-0.015
29	93162	8.7	2	50	10.519	10	27	54.17	0.0011	0.003
30	93175	8.5	2	51	37.939	11	6	24.76	0.0	0.0
31	93182	8.8	2	52	18.370	11	17	23.39	-0.0010	-0.027

TABLE 1e

GUIDING STAR	SAOSC	MAGN	RA			DEC			DATE
	110566	7.6	2^h	26^m	$55^s.228$	8°	$47'$	$2''.37$	1971.93336219

$MA = 0^s.0003$ $MD = 0''.017$ T= 21.93336219 T.Y.

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC '	DEC "	MA s	MD "
1	110574	8.6	2	27	19.623	9	53	33.69	-0.0026	-0.059
2	110597	9.3	2	29	34.243	9	55	0.51	0.0005	0.001
3	110601	8.7	2	30	3.794	9	37	34.20	0.0016	-0.002
4	110639	8.7	2	33	51.614	8	55	9.16	0.0051	-0.006
5	110568	8.4	2	27	4.682	8	16	36.16	0.0043	0.001
6	110576	8.6	2	27	32.642	8	39	23.13	0.0007	-0.022
7	110588	8.6	2	28	48.570	8	7	38.47	-0.0003	-0.013
8	110606	8.6	2	30	19.543	7	40	6.61	-0.0005	0.022
9	110609	8.7	2	30	35.620	8	9	30.26	0.0022	-0.013
10	110610	9.0	2	30	39.393	8	26	9.52	-0.0015	-0.012
11	110616	7.7	2	31	32.786	8	46	11.70	0.0020	0.022
12	110502	8.3	2	20	17.079	8	19	7.27	0.0014	-0.002
13	110520	8.8	2	22	30.612	8	19	32.73	-0.0035	-0.002
14	110525	8.7	2	22	58.059	8	29	49.49	0.0011	-0.010
15	110527	8.9	2	23	1.417	7	47	41.56	0.0003	-0.017
16	110539	8.6	2	25	5.608	8	38	4.48	0.0025	0.018
17	110543	4.3	2	25	29.791	8	14	13.15	0.0025	-0.004
18	110546	8.9	2	25	50.915	8	24	46.89	0.0014	-0.033
19	110548	9.1	2	25	56.644	8	3	39.22	0.0010	-0.058
20	110550	8.7	2	25	59.756	8	26	28.59	-0.0067	-0.085
21	110552	9.5	2	26	3.074	8	32	33.96	0.0029	-0.008
22	110554	8.8	2	26	6.879	7	30	14.48	0.0001	0.017
23	110556	8.5	2	26	14.767	8	32	52.13	0.0037	0.001
24	110558	8.6	2	26	20.047	8	9	24.63	-0.0004	0.013
25	92973	8.9	2	26	43.671	10	9	12.40	0.0005	0.010
26	110515	8.8	2	21	39.111	9	3	24.20	0.0021	0.013
27	110516	7.6	2	21	50.025	9	29	13.41	0.0045	-0.050
28	110537	6.5	2	24	43.583	9	58	38.74	-0.0195	-0.200
29	110565	6.3	2	26	54.866	9	20	36.92	-0.0010	-0.016
30	92922	7.3	2	21	29.521	10	2	47.71	0.0026	-0.021

STOP 0

TABLE 2a

GUIDING STAR	SAOSC	MAGNITUD	RA	DEC	DATE
	110723	4.4	$2^h\ 42^m\ 14^s.518$	$9^\circ\ 54'\ 14''.36$	1971.8707670

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC '	DEC ''	X s	Y '	X IN MM	Y IN MM
1	93088	8.7	2	42	50.564	10	35	55.95	35.435	41.697	6.6263	31.1895
2	93091	8.8	2	43	13.802	10	10	46.35	58.352	16.539	10.9119	12.3711
3	93110	8.5	2	44	55.712	10	1	5.27	158.744	6.889	29.6851	5.1529
4	93136	8.8	2	47	43.876	10	25	24.43	323.996	31.344	60.5872	23.4454
5	93139	8.9	2	47	52.260	11	12	1.50	331.458	77.997	61.9827	58.3415
6	93150	8.5	2	49	15.043	11	6	8.36	412.868	72.214	77.2062	54.0164
7	93151	8.6	2	49	15.483	11	10	8.23	413.216	76.217	77.2713	57.0099
8	93152	8.7	2	49	23.008	10	35	26.82	421.351	41.511	78.7927	31.0504
9	93158	9.0	2	49	52.544	10	45	28.11	450.185	51.583	84.1846	38.5839
10	93161	8.8	2	50	10.172	10	27	56.09	467.942	34.064	87.5052	25.4800
11	93162	8.7	2	50	10.543	10	27	54.24	468.308	34.034	87.5736	25.4572
							****					
12	110737	8.9	2	43	43.751	9	11	30.18	88.096	-42.727	16.4739	-31.9600
13	110742	9.1	2	44	11.783	8	58	0.22	115.850	-56.221	21.6640	-42.0537
14	110743	9.0	2	44	18.699	8	32	28.20	122.842	-81.764	22.9715	-61.1596
15	110745	8.0	2	44	42.102	9	5	59.33	145.747	-48.223	27.2547	-36.0707
16	110752	8.7	2	45	12.578	8	43	0.83	176.051	-71.193	32.9216	-53.2520
17	110753	8.7	2	45	18.299	8	41	27.48	181.723	-72.746	33.9821	-54.4143
18	110758	9.0	2	45	28.528	8	41	40.72	191.836	-72.520	35.8734	-54.2452
							****					
19	110662	9.5	2	36	40.590	9	16	27.54	-329.645	-37.621	-61.6436	-28.1402
20	110674	9.0	2	37	55.603	9	17	25.61	-255.562	-36.717	-47.7901	-27.4642
21	110698	8.8	2	39	53.008	9	35	28.29	-139.538	-18.733	-26.0937	-14.0161
22	110702	9.7	2	40	39.368	9	20	12.20	-93.895	-34.024	-17.5583	-25.4500
23	110716	8.7	2	41	44.982	9	1	9.93	-29.174	-53.077	-5.4555	-39.7016
							****					
24	93034	6.9	2	36	21.682	10	25	21.51	-347.100	31.321	-64.9077	23.4284
25	93036	9.0	2	36	29.138	10	27	25.04	-339.727	33.373	-63.5289	24.9627
26	93050	8.7	2	38	5.888	10	8	13.43	-244.776	14.082	-45.7731	10.5332
27	93059	6.7	2	39	18.476	10	19	56.07	-173.200	25.745	-32.3885	19.2575
28	93067	6.3	2	39	47.104	10	31	46.48	-144.946	37.572	-27.1048	28.1040
29	93071	8.7	2	40	24.754	10	28	44.54	-107.940	34.524	-20.1848	25.8237
30	93078	8.9	2	41	14.474	10	2	47.40	-59.123	8.556	-11.0560	6.4001

TABLE 2b

GUIDING STAR	SAOSC	MAGNITUD	RA	DEC	DATE
	110693	8.8	$2^h\ 39^m\ 53^s.008$	$9^\circ\ 35'\ 28''.29$	1971.8870742

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	X s	Y ′	X IN MM	Y IN MM
1	110723	4.4	2	42	14.518	9	54	14.36	139.408	18.799	26.0693	14.0616
2	93071	8.7	2	40	24.754	10	23	44.54	31.221	53.277	5.8383	39.8509
3	93078	8.9	2	41	14.474	10	2	47.40	80.221	27.329	15.0012	20.4424
4	93088	8.7	2	42	50.565	10	35	55.95	174.564	60.519	32.6434	45.2682
5	93091	8.8	2	43	13.802	10	10	46.35	197.658	35.366	36.9620	26.4538
6	93110	8.5	2	44	55.712	10	1	5.27	298.143	25.759	55.7528	19.2681
							****					
7	110702	9.7	2	40	39.368	9	20	12.20	45.747	-15.265	8.5547	-11.4184
8	110716	8.7	2	41	44.982	9	1	9.93	110.597	-34.290	20.6817	-25.6486
9	110737	8.9	2	43	43.751	9	11	30.18	227.807	-23.893	42.5998	-17.8717
10	110742	9.1	2	44	11.783	8	58	0.22	255.657	-37.376	47.8078	-27.9570
11	110743	9.0	2	44	13.699	8	32	28.20	262.820	-62.914	49.1474	-47.0599
12	110745	8.0	2	44	42.102	9	5	59.33	285.507	-29.365	53.3897	-21.9649
13	110752	8.7	2	45	12.578	8	43	0.83	315.971	-52.323	59.0865	-39.1374
14	110753	8.7	2	45	18.299	8	41	27.48	321.654	-53.874	60.1492	-40.2980
							****					
15	110646	8.6	2	34	44.644	9	20	44.30	-304.323	-14.595	-56.9084	-10.9169
16	110650	8.7	2	35	5.830	8	44	42.13	-283.910	-50.660	-53.0911	-37.8940
17	110658	8.3	2	36	19.212	8	42	17.39	-211.375	-53.124	-39.5271	-39.7366
18	110662	9.5	2	36	40.590	9	16	27.54	-189.918	-18.959	-35.5146	-14.1815
19	110674	9.0	2	37	55.603	9	17	25.61	-115.869	-18.025	-21.6675	-13.4827
20	110686	7.8	2	38	43.607	8	18	51.40	-68.689	-76.621	-12.8449	-57.3127
							****					
21	110632	8.7	2	33	1.040	9	54	18.38	-405.950	19.096	-75.9126	14.2842
22	93026	8.8	2	34	7.899	10	19	19.05	-339.619	44.039	-63.5088	32.9414
23	93034	6.9	2	36	21.682	10	25	21.47	-207.876	49.962	-38.8728	37.3716
24	93036	9.0	2	36	29.138	10	27	25.04	-200.521	52.017	-37.4973	38.9089
25	93050	8.7	2	38	5.888	10	8	13.43	-105.455	32.771	-19.7200	24.5129
26	93059	6.7	2	39	18.476	10	19	56.07	-33.974	44.467	-6.3532	33.2616
27	93067	6.3	2	39	47.104	10	31	46.46	-5.806	56.308	-1.0857	42.1182

TABLE 2c

GUIDING STAR	SAOSC	MAGNITUD	RA			DEC			DATE
	110650	8.7	2^h	35^m	$5^s.830$	8°	$44'$	$42''.13$	1971.9009253

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	X s	Y ′	X IN MM	Y IN MM
1	110662	9.5	2	36	40.590	9	16	27.54	93.526	31.771	17.4894	23.7646
2	110674	9.0	2	37	55.603	9	17	25.61	167.562	32.767	31.3340	24.5101
3	110698	8.8	2	39	53.008	9	35	23.29	283.234	50.896	52.9647	38.0705
4	110702	9.7	2	40	39.369	9	20	12.20	329.199	35.664	61.5603	26.6770
5	110716	8.7	2	41	44.982	9	1	9.93	394.327	16.688	73.7392	12.4825
6	110653	6.5	2	35	21.222	7	28	46.95	15.264	-75.932	2.8544	-56.7969
7	110658	8.3	2	36	19.212	8	42	17.35	72.537	-2.406	13.5644	-1.7994
8	110663	8.5	2	36	43.763	8	7	49.56	96.956	-36.865	18.1307	-27.5752
9	110667	8.4	2	37	5.995	7	59	19.51	119.012	-45.362	22.2552	-33.9304
10	110670	9.2	2	37	22.351	7	33	5.41	135.370	-71.600	25.3142	-53.5570
11	110678	8.5	2	38	14.692	8	1	35.72	187.037	-43.064	34.9759	-32.2120
12	110686	7.8	2	38	43.607	8	18	51.40	215.511	-25.784	40.3006	-19.2866
13	110696	8.9	2	39	46.546	7	59	43.06	278.048	-44.888	51.9949	-33.5764
14	110606	8.6	2	30	19.532	7	40	7.09	-283.828	-64.493	-53.0757	-48.2407
15	110609	8.7	2	30	35.668	8	9	29.58	-267.475	-35.117	-50.0179	-26.2676
16	110610	9.0	2	30	39.960	8	26	9.26	-263.029	-18.455	-49.1864	-13.8041
17	110640	6.0	2	33	55.348	7	30	46.80	-69.398	-73.928	-12.9774	-55.2979
18	110643	9.5	2	34	30.272	8	16	0.82	-35.190	-28.688	-6.5805	-21.4583
19	110647	8.6	2	34	52.896	7	23	15.98	-12.831	-81.451	-2.3994	-60.9252
20	110648	8.6	2	34	55.382	7	20	14.40	-10.366	-84.479	-1.9385	-63.1903
21	110601	8.7	2	30	3.829	9	37	34.16	-297.830	53.008	-55.6943	39.6501
22	110616	7.7	2	31	32.830	8	46	12.18	-210.526	1.563	-39.3684	1.1692
23	110632	8.7	2	33	1.040	9	54	18.38	-122.959	69.638	-22.9933	52.0890
24	110639	8.7	2	33	51.726	8	55	9.03	-73.210	10.456	-13.6902	7.8211
25	110646	8.6	2	34	44.644	9	20	44.30	-20.907	36.038	-3.9095	26.9566

TABLE 2d

GUIDING STAR	SAOSC	MAGNITUD	RA	DEC	DATE
	93186	8.2	$2^h\ 52^m\ 41^s.123$	$10^\circ\ 15'\ 20''.85$	1971.8240538

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	X s	Y ′	X IN MM	Y IN MM
1	93198	8.7	2	54	14.401	10	26	41.01	91.734	11.350	17.1543	8.4900
2	93200	8.7	2	54	25.111	10	36	12.61	102.217	20.881	19.1145	15.6189
3	93204	8.6	2	54	39.729	10	16	51.39	116.705	1.532	21.8238	1.1456
4	93227	8.5	2	57	7.856	10	18	17.10	262.462	3.051	49.0805	2.2824
5	93228	8.0	2	57	22.376	10	27	31.36	276.618	12.304	51.7277	9.2031
6	93231	8.9	2	57	52.246	10	36	7.84	305.867	20.943	57.1971	15.6651
7	93232	6.2	2	58	1.276	10	40	23.90	314.676	25.221	58.8445	18.8653
8	93244	8.5	2	59	5.851	10	39	19.56	378.197	24.223	70.7229	18.1192
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9	93212	7.8	2	55	41.951	10	0	20.52	178.089	-14.955	33.3027	-11.1860
10	93223	8.3	2	56	35.499	10	0	42.55	230.831	-14.553	43.1653	-10.8855
11	93234	8.1	2	58	6.884	10	2	48.89	320.825	-12.357	59.9942	-9.2504
12	110840	8.4	2	52	51.086	9	36	45.89	9.824	-38.584	1.8371	-28.8609
13	110859	8.3	2	53	47.701	9	55	30.27	65.583	-19.836	12.2641	-14.8376
14	110862	8.0	2	54	18.727	9	57	42.88	96.135	-17.618	17.9772	-13.1784
15	110888	8.6	2	56	58.958	9	26	14.30	254.400	-49.015	47.5728	-36.6630
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16	110788	9.0	2	48	37.663	9	11	18.94	-240.402	-63.954	-44.9552	-47.8378
17	110809	8.7	2	50	12.732	8	57	20.16	-146.625	-77.994	-27.4188	-58.3395
18	110817	6.8	2	51	4.950	9	7	57.58	-94.973	-67.383	-17.7600	-50.4027
19	110824	8.6	2	51	43.489	9	34	13.74	-56.836	-41.116	-10.6284	-30.7544
20	110834	8.8	2	52	33.405	8	54	35.33	-7.627	-80.773	-1.4262	-60.4185
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21	93134	8.6	2	47	35.402	11	24	32.20	-299.787	69.363	-56.0602	51.8838
22	93136	8.8	2	47	43.876	10	25	24.43	-292.386	10.203	-54.6762	7.6316
23	93139	8.9	2	47	52.260	11	12	1.50	-283.439	56.827	-53.0030	42.5068
24	93150	8.5	2	49	15.043	11	6	8.36	-202.260	50.869	-37.8226	38.0497
25	93151	8.6	2	49	15.483	11	10	7.81	-201.785	54.861	-37.7339	41.0357
26	93152	8.7	2	49	23.008	10	35	26.82	-194.756	20.164	-36.4194	15.0828
27	93158	9.0	2	49	52.544	10	45	28.11	-165.630	30.169	-30.9729	22.5666
28	93161	8.8	2	50	10.172	10	27	56.09	-148.446	12.625	-27.7594	9.4431
29	93162	8.7	2	50	10.543	10	27	54.24	-148.082	12.593	-27.6913	9.4193
30	93175	8.5	2	51	37.939	11	6	24.76	-62.008	51.076	-11.5954	38.2047
31	93182	8.8	2	52	18.348	11	17	22.80	-22.338	62.040	-4.1771	46.4061

TABLE 2e

GUIDING STAR SAOSC	MAGNITUD	RA	DEC	DATE
110566	7.6	$2^h\ 26^m\ 55^s.235$	$8^\circ\ 47'\ 2''.74$	1971.93336219

A/A	SAOSC	MAGN	RA h	RA m	RA s	DEC °	DEC ′	DEC ″	X s	Y ′	X IN MM	Y IN MM
1	110574	8.6	2	27	19.566	9	53	32.40	23.974	66.503	4.4832	49.7446
2	110597	9.3	2	29	34.254	9	55	0.53	156.681	63.011	29.2993	50.8722
3	110601	8.7	2	30	3.829	9	37	34.16	185.970	50.581	34.7765	37.8342
4	110639	8.7	2	33	51.726	8	55	9.03	411.578	8.346	76.9650	6.2431
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5	110568	8.4	2	27	4.776	8	16	36.18	9.443	-30.443	1.7658	-22.7716
6	110576	8.6	2	27	32.657	8	39	22.65	36.997	-7.666	6.9184	-5.7344
7	110588	8.6	2	28	48.563	8	7	38.18	112.200	-39.395	20.9815	-29.4672
8	110606	8.6	2	30	19.532	7	40	7.09	202.523	-66.886	37.8718	-50.0305
9	110609	8.7	2	30	35.668	8	9	29.97	218.234	-37.436	40.8097	-28.0392
10	110610	9.0	2	30	39.960	8	26	9.26	222.317	-20.825	41.5733	-15.5771
11	110616	7.7	2	31	32.830	8	46	12.18	274.386	-0.737	51.3101	-0.5514
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12	110502	8.3	2	20	17.110	8	19	7.23	-394.056	-27.720	-73.6884	-20.7342
13	110520	8.8	2	22	30.535	8	19	32.69	-261.949	-27.410	-48.9845	-20.5029
14	110525	8.7	2	22	58.083	8	29	49.27	-234.574	-17.150	-43.8653	-12.8282
15	110527	8.9	2	23	1.424	7	47	41.19	-231.707	-59.298	-43.3292	-44.3552
16	110539	8.6	2	25	5.663	8	38	4.87	-108.332	-8.948	-20.2582	-6.6933
17	110543	4.3	2	25	29.846	8	14	13.06	-84.513	-32.820	-15.8039	-24.5491
18	110546	8.9	2	25	50.946	8	24	46.17	-63.599	-22.271	-11.8930	-16.6588
19	110548	9.1	2	25	56.666	8	3	37.95	-57.995	-43.411	-10.8451	-32.4716
20	110550	8.7	2	25	59.609	8	26	26.73	-55.024	-20.596	-10.2895	-15.4061
21	110552	9.5	2	26	3.138	8	32	33.78	-51.520	-14.479	-9.6342	-10.8304
22	110554	8.8	2	26	6.881	7	30	14.85	-47.951	-76.808	-8.9669	-57.4525
23	110556	8.5	2	26	14.848	8	32	52.15	-39.938	-14.174	-7.4685	-10.6025
24	110558	8.6	2	26	20.038	8	9	24.92	-34.842	-37.630	-6.5155	-28.1475
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25	92073	8.9	2	26	43.682	10	9	12.62	-11.375	82.180	-2.1271	61.4710
26	110515	8.8	2	21	39.157	9	3	24.49	-312.193	16.504	-58.3802	12.3448
27	110516	7.6	2	21	50.124	9	29	17.31	-301.007	42.383	-56.2883	31.7022
28	110537	6.5	2	24	43.155	9	58	34.35	-130.114	71.564	-24.3314	53.5300
29	110565	6.3	2	26	54.844	9	20	36.57	-0.385	33.565	-0.0721	25.1065
30	92922	7.3	2	21	29.578	10	2	47.25	-320.799	75.920	-59.9893	56.7880

STOP 0

TABLE 3a

A/A	SAOSC	X mm	Y mm	X-XO mm	Y-YO mm	
		DATE 1971.8707670	SAOSC(G.S.) 110723			
		503.2754	107.6327			MEASUR. CO-OR. OF G.S.
		515.5366	131.6091	12.3112	-16.0736	MEASUR. CO-OR. OF AST.
1	93083	509.9080	228.8731	6.6326	31.1304	
2	93091	514.1816	210.0408	10.9062	12.3581	
3	93110	532.9782	202.8260	29.7023	5.1433	
4	93136	563.8958	221.0939	60.5204	23.4112	
12	110737	519.7592	165.7089	15.4838	-31.9733	
13	110742	524.9430	155.6153	21.6676	-42.0674	
14	110743	526.2492	136.5082	22.9738	-51.1745	
15	110745	530.5349	161.5881	27.2595	-36.0946	
16	110752	536.1947	144.4188	32.9193	-53.2639	
17	110753	537.2579	143.2537	33.9825	-54.4190	
18	110758	539.1396	143.4175	35.8642	-54.2652	
19	110662	441.6195	169.5542	-61.6559	-28.1285	
20	110674	455.4640	170.2198	-47.8114	-27.4529	
21	110698	477.1684	133.6787	-26.1070	-14.0040	
22	110702	485.6851	172.2336	-17.5903	-25.4431	
23	110716	497.8112	157.9720	-5.4542	-39.7107	
24	93034	438.3525	221.1308	-64.9229	23.4481	
25	93036	439.7170	222.6749	-63.5584	24.9922	
26	93050	457.4869	208.2315	-45.7885	10.5488	
27	93059	470.8743	216.9451	-32.4011	19.2624	
28	93067	476.1630	225.7325	-27.1124	28.1098	
29	93071	483.0860	223.5138	-20.1894	25.8314	
30	93078	492.2136	204.0737	-11.0558	6.3910	

TABLE 3b

A/A	SAOSC	X mm	Y mm	X-X0 mm	Y-Y0 mm	
	DATE 1971.8870742		SAOSC(G.S.) 110698			
		503.7087	197.7975			MEASUR. CO-OR. OF G.S.
		497.2651	185.2732	-6.4436	-12.5243	MEASUR. CO-OR. OF AST.
2	93071	509.0809	237.6442	5.3722	39.8467	
3	93073	518.7287	218.2111	15.0200	20.4136	
4	93088	536.3888	243.0396	32.6801	45.2421	
5	93091	540.6918	224.2116	36.9831	26.4141	
6	93110	559.4961	217.0264	55.7874	19.2289	
7	110702	512.2395	186.3758	8.5308	-11.4217	
8	110715	524.3877	172.1177	20.6790	-25.6798	
9	110737	546.3178	179.8839	42.6091	-17.9136	
10	110742	551.5146	169.7959	47.8059	-28.0016	
11	110743	552.8490	150.6893	49.1403	-47.1082	
12	110745	557.1068	175.7736	53.3981	-22.0239	
15	110646	446.7785	186.9120	-56.9302	-10.8855	
16	110650	450.5632	159.9190	-53.1455	-37.8785	
17	110658	464.1415	158.0813	-39.5672	-39.7162	
18	110662	468.1793	183.6265	-35.5294	-14.1710	
19	110674	482.0214	184.3059	-21.6873	-13.4916	
20	110686	490.8310	140.4712	-12.8777	-57.3263	
22	93026	440.2054	230.7862	-63.5033	32.9887	
23	93034	464.8594	235.1855	-38.8493	37.3880	
24	93036	466.2263	236.7327	-37.4824	38.9352	
25	93050	483.9929	222.3131	-19.7158	24.5156	
26	93059	497.3746	231.0502	-6.3341	33.2527	
27	93067	502.6568	239.9053	-1.0519	42.1078	

TABLE 3c

A/A	SAOSC	X mm	Y mm	X-X0 mm	Y-Y0 mm	
		DATE 1971.9009253		SAOSC(G.S.) 110650		
		503.9182	197.5792			MEASUR. CO-OR. OF G.S.
		515.2073	215.7712	11.2891	18.1920	MEASUR. CO-OR. OF AST.
1	110662	521.4228	221.3425	17.5046	23.7633	
2	110674	535.2604	222.0811	31.3422	24.5019	
3	110693	556.8848	235.6521	52.9666	38.0729	
4	110702	565.4651	224.2702	61.5479	26.6910	
5	110653	506.3070	140.8074	2.3888	-56.7718	
7	110658	517.4981	135.7976	13.5799	-1.7816	
8	110663	522.0597	170.0030	13.1415	-27.5762	
9	110667	526.1975	163.6523	22.2793	-33.9269	
10	110670	529.2758	144.0179	25.3576	-53.5613	
11	110678	538.9163	165.3798	34.9981	-32.1994	
12	110685	544.2508	178.2984	40.3326	-19.2808	
13	110696	555.9478	164.0066	52.0296	-33.5726	
14	110605	450.8552	149.3577	-53.0630	-48.2215	
15	110600	453.9046	171.3398	-50.0136	-26.2394	
16	110610	454.7337	183.7955	-49.1845	-13.7837	
17	110640	490.9650	142.3113	-12.9532	-55.2679	
18	110643	497.3651	176.1391	-6.5531	-21.4401	
19	110647	501.5577	136.6780	-2.3605	-60.9012	
20	110643	502.0050	134.4050	-1.9132	-63.1742	
21	110601	448.2043	237.2502	-55.7139	39.6710	
22	110615	464.5505	198.7580	-39.3677	1.1788	
23	110632	480.9079	249.6581	-23.0103	52.0889	
24	110639	490.2264	205.4088	-13.6918	7.8296	
25	110645	500.0089	224.5343	-3.9093	26.9551	

TABLE 3d

A/A	SAOSC	DATE 1971.8240533 X mm	Y mm	SAOSC(G.S.) 93185 X-XO mm	Y-YO mm	
		502.7843	197.5385			MEASUR. CO-OR. OF G.S.
		533.7939	203.2277	31.0096	5.6892	MEASUR. CO-OR. OF AST.
1	93198	519.9640	206.0342	17.1797	8.4957	
2	93200	521.9271	213.1607	19.1428	15.6222	
3	93204	524.6175	198.6928	21.8332	1.1543	
4	93227	551.8831	199.8249	49.0988	2.2864	
5	93228	554.5305	206.7503	51.7462	9.2118	
6	93231	560.0030	213.2122	57.2187	15.6737	
7	93232	561.6519	216.4094	58.8676	18.8709	
9	93212	536.1061	186.3569	33.3218	-11.1816	
10	93223	545.9709	186.6574	43.1866	-10.8811	
11	93234	562.3017	188.3041	60.0174	-9.2344	
12	110840	504.6277	168.6594	1.8434	-28.8691	
13	110859	515.0547	182.6970	12.2704	-14.8415	
14	110862	520.7858	184.3726	18.0015	-13.1659	
15	110883	550.3820	160.8732	47.5977	-36.6653	
16	110788	457.8257	149.7116	-44.9586	-47.8269	
17	110809	475.3648	139.2073	-27.4195	-58.3312	
18	110817	485.0211	147.1367	-17.7632	-50.4018	
19	110824	492.1626	166.7690	-10.6217	-30.7695	
20	110834	501.3627	137.1353	-1.4216	-60.4032	
22	93135	448.1051	205.1649	-54.6792	7.6264	
23	93139	449.7729	240.0355	-53.0114	42.4970	
24	93150	464.9702	235.5811	-37.8141	38.0426	
25	93151	465.0536	238.5717	-37.7307	41.0332	
26	93152	466.3665	212.6228	-36.4178	15.0843	
27	93158	471.8154	220.1049	-30.9689	22.5664	
30	93175	491.2076	235.7485	-11.5767	38.2100	
31	93182	498.6260	243.9522	-4.1583	46.4137	

TABLE 3e

A/A	SAOSC	X (mm)	Y (mm)	X-XO (mm)	Y-YO (mm)	
		DATE 1971.93336219		SAOSC(G.S.) 110566		
		503.6538	197.9008			MEASUR. CO-OR. OF G.S.
		528.9653	203.6840	25.3115	5.7832	MEASUR. CO-OR. OF AST.
1	110574	508.1538	247.6533	4.5000	49.7575	
2	110597	532.9681	248.7862	29.3143	50.8854	
5	110563	505.4192	175.1334	1.7654	-22.7574	
6	110576	510.5763	192.1645	6.9225	-5.7363	
7	110583	524.6423	168.4346	20.9885	-29.4662	
8	110606	541.5445	147.8579	37.8907	-50.0329	
9	110609	544.4781	169.8688	40.8243	-28.0320	
10	110610	545.2467	182.3318	41.5929	-15.5690	
11	110615	554.9889	197.3454	51.3351	-0.5554	
12	110502	429.9490	177.1649	-73.7048	-20.7359	
13	110520	454.6514	177.4035	-49.0024	-20.4973	
14	110525	459.7625	185.0628	-43.8913	-12.8380	
15	110527	460.3017	153.5419	-43.3521	-44.3589	
16	110539	483.3859	191.1954	-20.2679	-6.7054	
18	110546	491.7540	181.2400	-11.8998	-16.6608	
19	110548	492.8030	165.4370	-10.8508	-32.4638	
20	110550	493.3658	182.4866	-10.2880	-15.4142	
21	110552	493.9974	187.0807	-9.6564	-10.8201	
22	110554	494.5749	140.4353	-8.9789	-57.4655	
23	110556	496.1753	187.2922	-7.4785	-10.6086	
24	110558	497.1364	169.7351	-6.5174	-28.1657	
25	92973	501.5213	259.3800	-2.1325	61.4792	
26	110515	445.2536	210.2374	-58.3952	12.3366	
27	110516	447.3364	229.5946	-56.3174	31.6938	
28	110537	479.3168	251.4389	-24.3370	53.5381	
29	110565	503.5967	223.0124	-0.0571	25.1116	

TABLE 4

DATE 1971.8707670 SAOSC(G.S.) 110723

A/A	SAOSC	RA h	m	s	DEC °	′	″	DRA s	DDEC ″
		2	43	21.271	9	32	45.40	(Asteroid)	
13	110742	2	44	11.803	8	58	0.21	0.0201	-0.0135
24	93034	2	36	21.713	10	25	21.48	0.0305	-0.0274
28	93067	2	39	47.096	10	31	46.42	-0.0082	-0.0575

RESULTS USING SYMMETRICAL DISTRIBUTION OF THE STARS

		2	43	21.264	9	32	45.49	(Asteroid)	
13	110742	2	44	11.791	8	58	0.38	0.0084	0.1580
24	93034	2	36	21.716	10	25	21.34	0.0344	-0.1690
28	93067	2	39	47.097	10	31	46.32	-0.0069	-0.1603

DATE 1971.8870742 SAOSC(G.S.) 110698

A/A	SAOSC	RA h	m	s	DEC °	′	″	DRA s	DDEC ″
		2	39	18.140	9	18	44.22	(Asteroid)	
4	93088	2	42	50.536	10	35	56.13	-0.0287	0.1837
9	110737	2	43	43.781	9	11	30.31	0.0296	0.1259
18	110662	2	36	40.618	9	16	27.40	0.0284	-0.1383

RESULTS USING SYMMETRICAL DISTRIBUTION OF THE STARS

		2	39	18.140	9	18	44.22	(Asteroid)	
4	93088	2	42	50.536	10	35	56.13	-0.0287	0.1837
9	110737	2	43	43.781	9	11	30.31	0.0296	0.1259
18	110662	2	36	40.618	9	16	27.40	0.0284	-0.1383

DATE 1971.9009253 (SAOSC(G.S.) 110650

A/A	SAOSC	RA h	m	s	DEC °	′	″	DRA s	DDEC ″
		2	36	6.961	9	9	0.58	(Asteroid)	
3	110698	2	39	53.004	9	35	28.75	-0.0037	0.4630
16	110610	2	30	39.948	8	26	9.37	-0.0121	0.1050
24	110639	2	33	51.705	8	55	8.84	-0.0208	-0.1920

RESULTS USING SYMMETRICAL DISTRIBUTION OF THE STARS

		2	36	6.966	9	9	0.62		
3	110698	2	39	53.014	9	35	28.74	0.0065	0.4473
16	110610	2	30	39.943	8	26	9.50	-0.0165	0.2431
24	110639	2	33	51.706	8	55	8.91	-0.0198	-0.1178

DATE 1971.8240538 SAOSC(G.S.) 93186

A/A	SAOSC	RA h	m	s	DEC °	'	"	DRA s	DDEC "
		2	55	29.597	10	22	54.06	(Asteroid)	
9	93212	2	55	41.957	10	0	20.42	0.0057	-0.0999
18	110817	2	51	4.924	9	7	57.43	-0.0263	-0.1521
27	93158	2	49	52.537	10	45	28.19	-0.0068	0.0768
	RESULTS USING SYMMETRICAL DISTRIBUTION OF THE STARS								
		2	55	29.599	10	22	54.19	(Asteroid)	
9	93212	2	55	41.957	10	0	20.51	0.0063	-0.0143
18	110817	2	51	4.925	9	7	57.40	-0.0246	-0.1825
27	93158	2	49	52.545	10	45	28.34	0.0008	0.2333

DATE 1971.93336219 SAOSC(G.S.) 110566

A/A	SAOSC	RA h	m	s	DEC °	'	"	DRA s	DDEC "
		2	29	12.176	8	54	44.74	(Asteroid)	
7	110588	2	28	48.560	8	7	38.30	-0.0025	0.1248
18	110546	2	25	50.937	8	24	46.17	-0.0092	-0.0046
28	110537	2	24	43.154	9	58	34.59	-0.0010	0.2363
	RESULTS USING SYMMETRICAL DISTRIBUTION OF THE STARS								
		2	29	12.171	8	54	44.47	(Asteroid)	
7	110588	2	28	48.561	8	7	38.05	-0.0022	-0.1284
18	110546	2	25	50.944	8	24	46.07	-0.0015	-0.0982
28	110537	2	24	43.158	9	58	34.56	0.0026	0.2067

TABLE 5

DA	MTOD(LMST)	MTOD(GMST)	GMST AT 0 UT	DIFFER.(MST)	DIFFER.(MSL.T)	MH	UT	DT	ET
	h m s	h m s	h m s	h m s	h m s			s	
14	4 17 30.00	4 28 45.50	3 29 45.819	0 58 59.681	0 58 50.016	11	14.040856668	42.07	14.041343590
20	3 37 45.00	3 49 0.50	3 53 25.151	0 -4 -24.651	23 55 36.072	11	19.996945273	42.07	19.997432195
25	5 22 40.00	5 33 55.50	4 13 7.928	1 20 47.572	1 20 34.336	11	25.055952963	42.07	25.056439885
28	1 41 30.00	1 52 45.50	2 22 44.378	0 -29 -58.878	23 30 5.034	10	27.979236501	41.92	27.979721587
7	2 29 30.00	2 40 45.50	5 0 26.593	-2 -19 -41.093	21 40 41.791	12	6.903261470	42.22	6.903750128

MONTH	UT*	ET*	TO	T
10	45.040856668242	45.041343589538	0.718707670350	-0.218707670350
10	50.996945273287	50.997432194583	0.718870742657	-0.218870742657
10	56.055952963271	56.056439884557	0.719009253701	-0.219009253701
10	27.979236501353	27.979721686538	0.718240538656	-0.218240538656
10	67.903261470114	67.903750127521	0.719333622269	-0.219333622269

ZO		Z		RA			RA*			DEC		
m	s	m	s	h	m	s	°	′	″	°	′	″
0	-33.611	0	-2.238	2.0	43.0	21.271	40	50	19.0650	9.0	32.0	45.400
0	-33.636	0	-2.240	2.0	39.0	18.140	39	49	32.1000	9.0	18.0	44.220
0	-33.657	0	-2.241	2.0	36.0	6.961	39	1	44.4150	9.0	9.0	0.580
0	-33.539	0	-2.233	2.0	55.0	29.597	43	52	23.9550	10.0	22.0	54.060
0	-33.707	0	-2.245	2.0	29.0	12.176	37	18	2.6400	8.0	54.0	44.740

TABLE 6a

P= 0.0413435895   1-P= 0.9586564105   E2= -0.0129383284   F2= -0.0058783202

GENERAL PRECESSION IN RA= 0^{m} $-35^{s}.8490$ LOCAL SIDER. TIME WRT 1950.0= 4^{h} 16^{m} $54^{s}.1510$ or 64° $13'$ $32''.2645$

X0= -0.6273867000	D02= 0.0001931000		Y0= -0.7019288000	D02= 0.0002146000		Z0= -0.3043764000	D02= 0.0000930000	
X1= -0.6137044000	D12= 0.0001890000		Y1= -0.7117795000	D12= 0.0002178000		Z1= -0.3086484000	D12= 0.0000948000	
(1-P)X0= -0.6014482818			(1-P)Y0= -0.6729085438			(1-P)Z0= -0.2917923871		
P*X1= -0.0253727428			P*Y1= -0.0294275195			P*Z1= -0.0127606328		
E2*DX02= -0.0000024984			E2*DY02= -0.0000027766			E2*DZ02= -0.0000012033		
F2*DX12= -0.0000013001			F2*DY12= -0.0000014982			F2*DZ12= -0.0000006521		
DX= -0.0000103054	TOPOCEN CORRECTION		DY= -0.0000213422	TOPOCEN CORRECTION		DZ= -0.0000353000	TOPOCEN CORRECT	
X= -0.6268351285			Y= -0.7023616802			Z= -0.3045901752		

LAMDA= 0.7460782869   MI= 0.6448758040   NI= 0.1658384372   TEST= 1.0000000000

TABLE 6b

```
                  P=  0.9974321946    1-P=  0.0025678054    E2= -0.0004279647    F2= -0.0008526411
                       m     s                                          h    m     s              o    '     "
GENERAL PRECESSION IN RA=  0  -35.8758       LOCAL SIDER. TIME WRT 1950.0=   3   37   9.1242   or   54   17   16.8635

    X0= -0.5571281000   D02=  0.0001714000   Y0= -0.7489702000   D02=  0.0002307000   Z0= -0.3247752000   D02=  0.0001001000
    X1= -0.5425443000   D12=  0.0001667000   Y1= -0.7576987000   D12=  0.0002338000   Z1= -0.3285598000   D12=  0.0001015000

(1-P)X0= -0.0014305966                  (1-P)Y0= -0.0019232097                   (1-P)Z0= -0.0008339595
   P*X1= -0.5411511518                     P*Y1= -0.7557530772                      P*Z1= -0.3277161224
E2*DX02= -0.0000000734                  E2*DY02= -0.0000000987                   E2*DZ02= -0.0000000428
F2*DX12= -0.0000001421                  F2*DY12= -0.0000001993                   F2*DZ12= -0.0000000865
     DX= -0.0000138340  TOPOCEN CORRECTION   DY= -0.0000192435  TOPOCEN CORRECTION   DZ= -0.0000353000  TOPOCEN CORRECT

      X= -0.5425957978                        Y= -0.7576958285                         Z= -0.3285855113

  LAMDA=  0.7578762258           MI=  0.6320121896              NI=  0.1618153841             TEST=  1.0000000000
```

TABLE 6c

```
             P=  0.0564398846    1-P=  0.9435601154    E2= -0.0172505291    F2= -0.0093765829

GENERAL PRECESSION IN RA=  0m -35s.8985        LOCAL SIDER. TIME WRT 1950.0=   5h  22m  4s.1015  or  80°  31'  1".5230

    X0= -0.4672209000   D02=  0.0001424000   Y0= -0.7977810000   D02=  0.0002464000   Z0= -0.3459377000   D02=  0.0001057000
    X1= -0.4517094000   D12=  0.0001374000   Y1= -0.8050677000   D12=  0.0002483000   Z1= -0.3490969000   D12=  0.0001077000

(1-P)X0= -0.4408510063                   (1-P)Y0= -0.7527543325                   (1-P)Z0= -0.3264130161
   P*X1= -0.0254944264                      P*Y1= -0.0454379281                      P*Z1= -0.0197029887
E2*DX02= -0.0000024565                   E2*DY02= -0.0000042505                   E2*DZ02= -0.0000018406
F2*DX12= -0.0000012884                   F2*DY12= -0.0000023282                   F2*DZ12= -0.0000010099
     DX= -0.0000039047  TOPOCEN CORRECTION    DY= -0.0000233761  TOPOCEN CORRECTION    DZ= -0.0000353000  TOPOCEN CORRECT

      X= -0.4663530822                         Y= -0.7982222154                         Z= -0.3461541554

  LAMDA=  0.7669421338          MI=  0.6217005866           NI=  0.1590224642           TEST=  1.0000000000
```

TABLE 6d

```
                 P=  0.9797216865     1-P=  0.0202783135     E2= -0.0033783291     F2= -0.0065552226

GENERAL PRECESSION IN RA=  0m -35s.7725        LOCAL SIDER. TIME WRT 1950.0=   1h  40m  54s.2275   or   25°  13'  33".4131

    X0= -0.8372295000    D02=  0.0002514000    Y0= -0.4914175000    D02=  0.0001494000    Z0= -0.2130900000    D02=  0.0000547000
    X1= -0.8275451000    D12=  0.0002484000    Y1= -0.5045793000    D12=  0.0001533000    Z1= -0.2187967000    D12=  0.0000665000

(1-P)X0= -0.0169776022                 (1-P)Y0= -0.0099651181                         (1-P)Z0= -0.0043211058
   P*X1= -0.8107638811                    P*Y1= -0.4943472828                            P*Z1= -0.2143598719
E2*DX02= -0.0000008493                 E2*DY02= -0.0000005047                         E2*DZ02= -0.0000002186
F2*DX12= -0.0000016283                 F2*DY12= -0.0000010049                         F2*DZ12= -0.0000004359
     DX= -0.0000214398  TOPOCEN CORRECTION  DY= -0.0000101007  TOPOCEN CORRECTION       DZ= -0.0000353000  TOPOCEN CORRECT

      X= -0.8277654008                       Y= -0.5043240112                             Z= -0.2187169322

  LAMDA=  0.7090725757          MI=  0.6817201390              NI=  0.1802047019              TEST=  1.0000000000
```

TABLE 6e

```
        P=  0.9037501275    1-P=  0.0962498725    E2= -0.0158930350    F2= -0.0275998823

GENERAL PRECESSION IN RA=  0m -35s.9516        LOCAL SIDER. TIME WRT 1950.0=   2h  28m  54s.0484   or   37°  13'  30".7254

   X0= -0.2898213000    D02=  0.0000904000    Y0= -0.8639732000    D02=  0.0002660000    Z0= -0.3746416000    D02=  0.0001153000
   X1= -0.2730579000    D12=  0.0000858000    Y1= -0.8684294000    D12=  0.0002676000    Z1= -0.3765747000    D12=  0.0001161000

(1-P)X0= -0.0278952632                    (1-P)Y0= -0.0831573103                    (1-P)Z0= -0.0360592062
   P*X1= -0.2467761119                       P*Y1= -0.7848431810                       P*Z1= -0.3403294331
E2*DX02= -0.0000014367                    E2*DY02= -0.0000042275                    E2*DZ02= -0.0000018325
F2*DX12= -0.0000023581                    F2*DY12= -0.0000073857                    F2*DZ12= -0.0000032043
     DX= -0.0000188715  TOPOCEN CORRECTION     DY= -0.0000143373  TOPOCEN CORRECTION     DZ= -0.0000353000  TOPOCEN CORRECT

      X= -0.2746940514                          Y= -0.8680264419                          Z= -0.3764289762

  LAMDA=  0.7858614982           MI=  0.5986819274              NI=  0.1549246767              TEST=  1.0000000000
```

TABLE 7a

T1= 0.0870255509 T2= 0.1894827764 T3= 0.1024572255

P2	R2
2.3190543576	3.2755845365
2.4478499690	3.4038782071
2.4552741439	3.4112746017
2.4556685960	3.4116675812
2.4556894581	3.4116883655
2.4556905612	3.4116894645
2.4556906196	3.4116895226
2.4556906226	3.4116895256
2.4556906228	3.4116895258
2.4556906228	3.4116895258

P1= 2.4317902733 P2= 2.4556906228 P3= 2.4839625653 CONTROL BY 3RD EQUAT= 0.0000000000

R1=	3.4081840523	CO-ORDIN. OF R1=(	2.4411410496	2.2705643879	0.7078744737)
R2=	3.4116895258	CO-ORDIN. OF R2=(	2.4037053388	2.3097222360	0.7259540325)
R3=	3.4147061793	CO-ORDIN. OF R3=(	2.3714086323	2.3425031993	0.7411600034)

C1= 0.4593340887 C3= 0.5407781780

TABLE 7b

T1= C.C870227420 T2= C.1894775930 T3= C.1024548510

B1= C.CC09866617 B2= C.C037348C5C B3= C.CC04993219

1+B1*R1 **-3= 1.CCC0249229 1-B2*R2**-3= 0.9999C59499 1+B3*R3**-3= 1.CCCC125407

C1= 0.4593318659 C3= C.5407804224

N1= 0.CC21700659 N3= C.CC22889893

P2	R2
2.3189056C63	3.2754363877
2.4477138C20	3.40374255C8
2.4551412236	3.4111421778
2.4555359772	3.4115354575
2.455556862C	3.4115562644
2.4555579667	3.4115573649
2.4555580251	3.4115574231
2.4555580282	3.4115574262
2.4555580284	3.4115574264
2.4555580284	3.4115574264

P1= 2.4316468477 P2= 2.4555580284 P3= 2.4838389573 CCNTROL BY 3RD EQUAT= C.CCCCCCCC00

R1= 3.4C80408488 CO-ORDIN. OF R1=(2.4410340429 2.2704718962 C.7078506883)

R2= 3.4115574264 CO-ORDIN. OF R2=(2.4C36048487 2.3096384347 C.7259325767)

R3= 3.4145833595 CO-ORDIN. OF R3=(2.3713138321 2.3424263520 C.7411403469)

C1= C.4593318722 C3= 0.5407804291

TABLE 7c

```
          T1=  C.0870227411             T2=  0.1894775910             T3=  C.1024548499
          B1=  C.CCC9866617             B2=  0.0037348049             B3=  C.CC04993219
1+B1*R1 **-3=  1.CCC0249261   1-B2*R2**-3=  0.9999059389   1+B3*R3**-3=  1.CCC0125420
                      C1=  0.4593318724          C3=  0.5407804289
                      N1=  C.CC217CC704          N3=  C.0022889870

                         P2                        R2
                     2.3189055251             3.2754363068
                     2.44771377C8             3.4037425197
                     2.455141199C             3.4111421533
                     2.4555359531             3.4115354335
                     2.4555568380             3.41155624C4
                     2.4555579427             3.411557341C
                     2.4555580011             3.4115573992
                     2.4555580042             3.4115574023
                     2.4555580043             3.4115574024
                     2.4555580C44             3.4115574024

P1=  2.4316468245    P2=  2.4555580C44    P3=  2.4838389321    CCNTRCL BY 3RD EQUAT=  0.CCCCCCCCCO

   R1=  3.4080408257      CO-ORDIN. OF R1=(  2.4410340256    2.2704718813    C.7C78506844 )
   R2=  3.4115574024      CO-ORDIN. OF R2=(  2.4036048305    2.3096384195    C.7259325728 )
   R3=  3.4145933345      CO-ORDIN. OF R3=(  2.3713138128    2.3424263364    C.7411403429 )

                      C1=  0.4593318724          C3=  0.5407804289
```

TABLE 7d

```
              T1=  0.0870227411                 T2=  0.1804775910               T3=  0.1024548499
              B1=  0.0009866617                 B2=  0.0037348049               B3=  0.0004993219
1+B1*R1 **-3=  1.0000249261    1-B2*R2**-3=  0.9999059389    1+B3*R3**-3=  1.0000125420
                         C1=  0.4593318724            C3=  0.5407804289
                         N1=  0.0021700704            N3=  0.0022889870


                          P2                         R2
                    2.3189055251               3.2754363068
                    2.4477137708               3.4037425197
                    2.4551411991               3.4111421533
                    2.4555359531               3.4115354335
                    2.4555568380               3.4115562404
                    2.4555579427               3.4115573410
                    2.4555580011               3.4115573992
                    2.4555580042               3.4115574023
                    2.4555580044               3.4115574024
                    2.4555580044               3.4115574024


P1=  2.4316468245     P2=  2.4555580044     P3=  2.4838389321     CONTROL BY 3RD EQUAT=  0.0000000000

   R1=  3.4080408257      CO-ORDIN. OF R1=(  2.4410340256      2.2704718813      0.7078506844 )
   R2=  3.4115574024      CO-ORDIN. OF R2=(  2.4036048305      2.3096384195      0.7259325728 )
   R3=  3.4145833345      CO-ORDIN. OF R3=(  2.3713138128      2.3424263364      0.7411403430 )

                         C1=  0.4593318724          C3=  0.5407804289
```

TABLE 8

K1=	4.8266887370	K2=	4.8237461520	K3=	4.8220150392
M1=	0.0000238107	M2=	0.0001130882	M3=	0.0000331005
L1=	0.0000126036	L2=	0.0000598764	L3=	0.0000175355
H1=	0.0000285724	H2=	0.0001356961	H3=	0.0000397197
Y1=	1.0000317460	Y2=	1.0001507485	Y3=	1.0000441309
X1=	0.0000112056	X2=	0.0000531777	X3=	0.0000155621
KS1=	0.0000000000	KS2=	0.0000000002	KS3=	0.0000000000
H1=	0.0000285724	H2=	0.0001356961	H3=	0.0000397197
Y1=	1.0000317460	Y2=	1.0001507485	Y3=	1.0000441309

C1= 0.4593318726 C3= 0.5407804288

TABLE 9a

```
    PARAMETER OF ORBIT IN AU=   3.5993471542
             ECCENTRICITY=   0.0856201377
      SEMI-MAJOR AXIS IN AU=   3.6259281371
   MEAN DAILY MOTION IN SEC= 513".8990251332
   ECCENTRIC ANOMALY AT T1=  45°  25'  31".272
   ECCENTRIC ANOMALY AT T3=  47°   5'  47".910
        MEAN ANOMALY AT T1=  41°  55'  51".106
        MEAN ANOMALY AT T3=  43°  30'  11".598
TEST FOR ECCENTRIC ANOMALIES=   0.0000000000
TIME OF PERIHELION PASSAGE=1971 OCTOB.-248.7095996920
            VECTOR ELEMENTS
                       PX=   0.9939709540
                       PY=  -0.0549672790
                       PZ=  -0.0948701261
                       QX=   0.0855459647
                       QY=   0.9300114809
                       QZ=   0.3574360550
        TEST FOR PX,PY,PZ=  -0.0000000000
        TEST FOR QX,QY,QZ=   0.0000000000
TEST FOR TIME OF PER. PAS. IN SEC=  -0.0001029789
              INCLINATION=   4°  26'  59".743
  LONGITUDE OF PERIHELION= 355°   0'  33".879
        TRUE ANOMALY AT T1=  49°   2'   1".429
        TRUE ANOMALY AT T3=  50°  48'  13".138
```

C1= 0.4593318726 C3= 0.5407804288

TABLE 9b

PARAMETER OF ORBIT IN AU=	3.5993471609
ECCENTRICITY=	0.0856201276
SEMI-MAJOR AXIS IN AU=	3.6259281375
MEAN DAILY MOTION IN SEC=	513".8990250537
ECCENTRIC ANOMALY AT T1=	45° 25′ 31".254
ECCENTRIC ANOMALY AT T3=	47° 5′ 47".892
MEAN ANOMALY AT T1=	41° 55′ 51".091
MEAN ANOMALY AT T3=	43° 30′ 11".583
TEST FOR ECCENTRIC ANOMALIES=	-0.0000000000
TIME OF PERIHELION PASSAGE=1971 OCTOB.-	248.7095703038
VECTOR ELEMENTS	
PX=	0.9939709624
PY=	-0.0549671873
PZ=	-0.0948700914
QX=	0.0855458669
QY=	0.9300114862
QZ=	0.3574360646
TEST FOR PX,PY,PZ=	-0.0000000000
TEST FOR QX,QY,QZ=	0.0000000000
TEST FOR TIME OF PER. PAS. IN SEC=	-0.0001029838
INCLINATION=	4° 26′ 59".743
LONGITUDE OF PERIHELION=	355° 0′ 33".899
TRUE ANOMALY AT T1=	49° 2′ 1".409
TRUE ANOMALY AT T3=	50° 48′ 13".118

TABLE 10a

COEFFICENTS OF THE SYSTEM FOR LAMDA

-0.1024572255D 00	0.5248741533D-02	-0.1792571650D-03	0.4591547946D-05	0.1179793897D-01
0.8702555087D-01	0.3786723252D-02	0.1098472257D-03	0.2389878832D-05	-0.9065907942D-02
-0.3959529338D 00	0.7838936288D-01	-0.1034616507D-01	0.1024148702D-02	0.4880365017D-01
0.2908241540D 00	0.4228934426D-01	0.4099587589D-02	0.2980647730D-03	-0.2798527239D-01

DERIVATIVES OF LAMDA

0.1092052282	-0.1186800122	0.0219747054	3.8735437667

COEFICENTS OF THE SYSTEM FOR MI

-0.1024572255D 00	0.5248741533D-02	-0.1792571650D-03	0.4591547946D-05	-0.1286361442D-01
0.8702555087D-01	0.3786723252D-02	0.1098472257D-03	0.2389878832D-05	0.1031160303D-01
-0.3959529338D 00	0.7838936288D-01	-0.1034616607D-01	0.1024148702D-02	-0.4970794937D-01
0.2908241540D 00	0.4228934426D-01	0.4099587589D-02	0.2980647730D-03	0.3333026223D-01

DERIVATIVES OF MI

-0.1217862263	0.0779301828	0.0193837956	-4.3177456874

COEFICENTS OF THE SYSTEM FOR NI

-0.1024572255D 00	0.5248741533D-02	-0.1792571650D-03	0.4591547946D-05	-0.4023053148D-02
0.8702555087D-01	0.3786723252D-02	0.1098472257D-03	0.2389878832D-05	0.2792919904D-02
-0.3959529338D 00	0.7838936288D-01	-0.1034616607D-01	0.1024148702D-02	-0.1838931780D-01
0.2908241540D 00	0.4228934426D-01	0.4099587589D-02	0.2980647730D-03	0.6390707339D-02

DERIVATIVES OF NI

-0.0355359924	0.0767200078	0.0974547762	-0.6708126409

TABLE 10b

COEFFICENTS OF THE SYSTEM FOR LAMDA

-0.1024542772D 00	0.5248439461D-02	-0.1792416905D-03	0.4591019462D-05	0.1179793897D-01
0.8702304662D-01	0.3786505321D-02	0.1098377430D-03	0.2389603758D-05	-0.9065907942D-02
-0.3959415398D 00	0.7838485147D-01	-0.1034527293D-01	0.1024030823D-02	0.4880365017D-01
0.2908157852D 00	0.4228691046D-01	0.4099233689D-02	0.2980304660D-03	-0.2798527239D-01

DERIVATIVES OF LAMDA

0.1092083708	-0.1186868428	0.0219766036	3.8739896503

COEFICENTS OF THE SYSTEM FOR MI

-0.1024542772D 00	0.5248439461D-02	-0.1792416905D-03	0.4591019462D-05	-0.1286361442D-01
0.8702304662D-01	0.3786505321D-02	0.1098377430D-03	0.2389603758D-05	0.1031160303D-01
-0.3959415398D 00	0.7838485147D-01	-0.1034527293D-01	0.1024030823D-02	-0.4970794937D-01
0.2908157852D 00	0.4228691046D-01	0.4099233689D-02	0.2980304660D-03	0.3330262223D-01

DERIVATIVES OF MI

-0.1217897310	0.0779346680	0.0193854691	-4.3182427142

COEFICENTS OF THE SYSTEM FOR NI

-0.1024542772D 00	0.5248439461D-02	-0.1792416905D-03	0.4591019462D-05	-0.4023053148D-02
0.8702304662D-01	0.3786505321D-02	0.1098377430D-03	0.2389603758D-05	0.2792919904D-02
-0.3959415398D 00	0.7838485147D-01	-0.1034527293D-01	0.1024030823D-02	-0.1838931780D-01
0.2908157852D 00	0.4228691046D-01	0.4099233689D-02	0.2980304660D-03	0.5890707339D-02

DERIVATIVES OF NI

-0.0355370151	0.0757244234	0.0974631898	-0.6708898599

TABLE 11a

D= -0.0085737521 D1= -0.0207046313 D2= -0.0049134399

RO	PO
3.2607642859	2.3041735634
3.3890834856	2.4329993137
3.3966706494	2.4405152232
3.3970838759	2.4410300116
3.3971062759	2.4410524962
3.3971074898	2.4410537148
3.3971075556	2.4410537808
3.3971075592	2.4410537844
3.3971075594	2.4410537846
3.3971075594	2.4410537846

T1= -0.1024572255 T2= 0.0870255509

RO= 3.3971075594 PO= 2.4410537846 POD= 0.2895446400

TABLE 11b

D= -0.0085744923 D1= -0.0207052271 D2= -0.0049137227

RO	PO
3.2606322197	2.3040409552
3.3889355616	2.4323508280
3.3965226249	2.4404655391
3.3969358941	2.4408814705
3.3969582990	2.4409039601
3.3969595134	2.4409051790
3.3969595792	2.4409052451
3.3969595828	2.4409052487
3.3969595830	2.4409052489
3.3969595830	2.4409052489

T1= -0.1024542772 T2= 0.0870230466

RO= 3.3969595830 PO= 2.4409052489 POD= 0.2896353500

VX	VY	VZ
0.8513106812	-0.4998623236	-0.2167338797
0.8526190684	-0.5007068654	-0.2171000662
0.8526191209	-0.5007067610	-0.2171000209
0.8526191206	-0.5007067615	-0.2171000211

TABLE 12a

ANGULAR MOMENTUM PER UNIT MASS=	0.0327144009
PARAMETER OF ORBIT=	3.6167214268
ECCENTRICITY=	0.1017724504
SEMI-MAJOR AXIS=	3.6545741598
MEAN DAILY MOTION=	507″.3686599942
ECCENTRIC ANOMALY AT T2=	46° 9′ 39″.861
MEAN ANOMALY AT T2=	41° 57′ 18″.491
TIME OF PERIHELION PASSAGE=	1971 OCT08.-246.4007949435
INCLINATION=	4° 24′ 58″.794
LONGITUDE OF PERIHELION=	354° 29′ 39″.821
TRUE ANOMALY AT T2=	50° 31′ 49″.113

TABLE 12b

ANGULAR MOMENTUM PER UNIT MASS=	0.0327130456
PARAMETER OF ORBIT=	3.6164217803
ECCENTRICITY=	0.1014912555
SEMI-MAJOR AXIS=	3.6540603373
MEAN DAILY MOTION=	507".9757861147
ECCENTRIC ANOMALY AT T2=	46° 6' 39".285
MEAN ANOMALY AT T2=	41° 55' 12".450
TIME OF PERIHELION PASSAGE=	1971 OCTOB.-246.0899543072
INCLINATION=	4° 24' 54".649
LONGITUDE OF PERIHELION=	354° 33' 38".849
TRUE ANOMALY AT T2=	50° 27' 50".643

TABLE 13

X	Y	Z
0.2274998773	-0.3149457117	-0.1920250289
0.1130161927	-0.6523456838	-0.3011631125
-0.6273867000	-0.7019288000	-0.3043754000
1.3513716839	0.3837744155	0.1398889233
-1.3713848860	-4.7290200405	-1.9954407553
4.2196437833	7.4884537959	2.9141278533
-17.8009185607	-4.1940954449	-1.5865002988
-14.0294204168	-24.9932138840	-9.8861673093
2.4412041126	2.2702909762	0.7077673727
0.3189456434	-0.2047799301	-0.1424231127
0.2299592237	-0.6238829608	-0.2957193985
-0.5425443000	-0.7576987000	-0.3285598000
1.3279008667	0.4623310824	0.1765808202
-1.3280906775	-4.7372810920	-2.0000375196
4.1881924295	7.5022642976	2.9211876171
-17.7952686682	-4.2161457303	-1.5962671380
-14.0128426261	-25.0010305525	-9.8897745566
2.4034986666	2.3097477130	0.7259831568
0.3568822004	-0.0879134149	-0.0837360264
0.3227003821	-0.5869769185	-0.2849443817
-0.4672209000	-0.7977810000	-0.3459377000
1.3045411675	0.5264849803	0.2066581709
-1.2919335927	-4.7439061716	-2.0037690436
4.1619423619	7.5136845426	2.9270508024
-17.7905419921	-4.2345305901	-1.6044304186
-13.9989829134	-25.0075299020	-9.8928269635
2.3710432317	2.3426973118	0.7412663015

TABLE 14

VELOCITIES OF THE PLANETS

VX	VY	VZ
0.674229922440345519	1.24340916807981605	0.596611165705761753
1.10741042893800157	0.360746153132148584	0.0926312316198701496
-0.852644829972549895	0.500442733394442354	0.216972148563807060
-0.251703919455896930	0.753162100207822038	0.352501012201820479
0.420001820940074766	-0.0783732513728220559	-0.0439530210773910971
-0.304955616121567213	0.133217626857053781	0.0683401001313072159
0.0548306703628411744	-0.213762473478564016	-0.0949134509713257662
0.161118020193995876	-0.0755889966201690466	-0.0354966747839541002
-0.368434268679320082	0.379299386151404072	0.175549777167255047

INITIAL TOTAL ENERGY = -0.33213347D-07

		X	Y	Z	C	RA	DEC
		0.24004174D+01	0.23129154D+01	0.72744956D+00	-0.33213349D-07	2 38 58.467	9 17 43.16
1971	Nov. 21	0.23972418D+01	0.23161713D+01	0.72895745D+00	-0.33213352D-07	2 38 39.006	9 16 41.01
		0.23940617D+01	0.23194229D+01	0.73046397D+00	-0.33213354D-07	2 38 19.695	9 15 40.08
	Nov. 22	0.23908771D+01	0.23226702D+01	0.73196914D+00	-0.33213356D-07	2 38 0.539	9 14 40.38
		0.23876881D+01	0.23259132D+01	0.73347294D+00	-0.33213358D-07	2 37 41.541	9 13 41.93
	Nov. 23	0.23844947D+01	0.23291518D+01	0.73497538D+00	-0.33213360D-07	2 37 22.706	9 12 44.75

		X	Y	Z	C	RA			DEC		
		0.23812968D+01	0.23323861D+01	0.73647646D+00	-0.33213363D-07	2	37	4.039	9	11	48.85
Nov.	24	0.23780945D+01	0.23356161D+01	0.73797616D+00	-0.33213365D-07	2	36	45.543	9	10	54.25
		0.23748878D+01	0.23388417D+01	0.73947450D+00	-0.33213367D-07	2	36	27.224	9	10	0.97
	25	0.23716766D+01	0.23420630D+01	0.74097146D+00	-0.33213370D-07	2	36	9.084	9	9	9.01
		0.23684611D+01	0.23452800D+01	0.74246704D+00	-0.33213372D-07	2	35	51.128	9	8	18.40
	26	0.23652411D+01	0.23484925D+01	0.74396125D+00	-0.33213375D-07	2	35	33.360	9	7	29.15
		0.23620168D+01	0.23517008D+01	0.74545407D+00	-0.33213377D-07	2	35	15.784	9	6	41.27
	27	0.23587880D+01	0.23549046D+01	0.74694552D+00	-0.33213380D-07	2	34	58.404	9	5	54.78
		0.23555549D+01	0.23581041D+01	0.74843557D+00	-0.33213383D-07	2	34	41.223	9	5	9.69
	28	0.23523174D+01	0.23612992D+01	0.74992424D+00	-0.33213385D-07	2	34	24.246	9	4	26.01
		0.23490756D+01	0.23644899D+01	0.75141152D+00	-0.33213388D-07	2	34	7.475	9	3	43.76
	29	0.23458294D+01	0.23676763D+01	0.75289740D+00	-0.33213391D-07	2	33	50.914	9	3	2.94
		0.23425788D+01	0.23708583D+01	0.75438189D+00	-0.33213393D-07	2	33	34.568	9	2	23.57
	30	0.23393239D+01	0.23740358D+01	0.75586498D+00	-0.33213396D-07	2	33	18.438	9	1	45.66
		0.23360647D+01	0.23772090D+01	0.75734668D+00	-0.33213399D-07	2	33	2.529	9	1	9.23
Dec.	1	0.23328011D+01	0.23803778D+01	0.75882697D+00	-0.33213402D-07	2	32	46.844	9	0	34.27
		0.23295333D+01	0.23835421D+01	0.76030585D+00	-0.33213405D-07	2	32	31.386	9	0	0.81
Dec.	2	0.23262611D+01	0.23867021D+01	0.76178333D+00	-0.33213407D-07	2	32	16.158	9	59	28.84

APPENDIX A-1

Proof of the convergence of the series (3.37) (Chapter 3)

The Functions

$$\alpha_i=\alpha_i(\tau),\ \beta_{ijk}=\beta_{ijk}(\tau),\ \gamma_{ijk}=\gamma_{ijk}(\tau),\ \delta_{ijk}=\delta_{ijk}(\tau),\ x_i=x_i(\tau),\ x_i'=x_i'(\tau)\ (i,j,k=1,\dots n)$$

$$\alpha_{ij}=\alpha_{ij}(\tau),\quad b_{ij}=b_{ij}(\tau)\quad (i,j=1,\dots n\ i\neq j), \tag{A-1.1}$$

possess first order derivatives with respect to τ in any closed interval $[-\tau_1, \tau_1]$, as was mentioned previously. Hence, according to the theorem 1 (Chapter 3) the functions (A-1.1) are continuous in this interval, which is closed and obviously bounded. Therefore, the functions (A-1.1) are bounded in this range. (H. S. W. Massey, p 49-50)

Let us take

$$B = \max \left\{ \begin{array}{l} \text{absolute values of the upper and lower bounds} \\ \text{of the functions (A-1.1) in the interval } [-\tau_1, \tau_1] \end{array} \right\},$$

and

$$A = \max\left\{ B, (1+m_i), m_i \quad (i=1,\dots n) \right\}. \tag{A-1.2}$$

The right hand side of the relation

$$x_i''=\left[-(1+m_i)\alpha_i-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\alpha_{ij}\right]x_i+\sum_{\substack{j=1\\ j\neq i}}^{n} m_j(\alpha_{ij}-\alpha_j)x_j \quad (i=1,\dots n) \tag{A-1.3}$$

contains $(3n-2)$ products, with three factors in each one.

Interpreting $F_1, F_2, \dots F_v, \dots$ as representing any one of the functions (A-1.1) and the factors $(1+m_i)$, m_i, $(i=1,\dots n)$, the products which appear in the relation (A-1.3) have the general form

$$F_1\cdot F_2\cdot F_3 \ . \tag{A-1.4}$$

Clearly, in the interval $[-\tau_1, \tau_1]$, the following general relation is valid

$$|F_1\cdot F_2\cdot F_3| \leqslant A^3,$$

hence

$$x_i''(\tau) \leqslant (3n-2)A^3 \equiv \alpha_2 \quad (i=1,\ldots n) \quad \tau\in[-\tau_1,\tau_1] \ . \tag{A-1.5}$$

Since the second order derivative $x_i''(\tau) \quad (i=1,\ldots n)$ contains $(3n-2)$ products of the form (A-1.4), the third order derivative $x_i^{(3)}(\tau)$ $(i=1,\ldots n)$ will contain $(3n-2)\cdot 3$ products of the general form

$$F_1\cdot F_2\cdot F_3' \ , \tag{A-1.6}$$

where $F_3' \equiv dF_3/d\tau$ and the meaning of F_1, F_2, F_3 is the same as above.

We define,

$\theta \equiv$ maximum number of products which appear in the right members of the relations (3.26),

$\mu \equiv$ maximum number of factors of these products .

From the equations (3.26), (A-1.3) and the definition of F_3 it is evident that the derivative F_3' will be either zero or a sum of θ products, at most, and every one of them will contain, at most, μ factors.

It is obvious now that the function $x_i^{(3)}(\tau)$ will be represented by a sum of $(3n-2)\cdot 3\cdot\theta$ products, at most, of the general form

$$F_1\cdot F_2\cdots F_\nu \ ,$$

where $\nu \leqslant \mu+2$.

In the interval $[-\tau_1, \tau_1]$ the relation

$$|F_1\cdot F_2\cdots F_\nu| \leqslant A^{\mu+2}$$

is valid, since $\nu \leqslant \mu+2$. Hence

$$|x_i^{(3)}(\tau)| \leqslant (3n-2)\,3\,\theta A^{\mu+2} \equiv \alpha_3 \quad (i=1,\ldots n) \quad \tau\in[-\tau_1,\tau_1] .$$

Working as above we obtain the relations

$$\begin{aligned}
|x_i^{(4)}(\tau)| &\leqslant (3n-2)\,3\,\theta^2(\mu+2)A^{2\mu+1} \equiv \alpha_4 \ , \\
|x_i^{(5)}(\tau)| &\leqslant (3n-2)\,3\,\theta^3(\mu+2)(2\mu+1)A^{3\mu} \equiv \alpha_5 \ , \\
&\vdots \\
|x_i^{(\nu)}(\tau)| &\leqslant (3n-2)\,3\,\theta^{\nu-2}(\mu+2)(2\mu+1)\cdots(\nu(\mu-1)-3\mu+6)A^{\nu(\mu-1)-2\mu+5} \equiv \alpha_\nu \ , \\
&\vdots \qquad (i=1,\ldots n) \quad \tau\in[-\tau_1,\tau_1] .
\end{aligned} \tag{A-1.7}$$

At the point $\tau = 0$ we have $x_{i\,\tau=0}^{(\nu)} \equiv x_{io}^{\nu}$ $(i=1,\dots n)$ $\nu \geqslant 0$, hence

$$|x_{io}| \leqslant A ,$$

$$|x'_{io}\tau| \leqslant A|\tau| , \tag{A-1.8}$$

$$\left|x''_{io}\frac{\tau^2}{2!}\right| \leqslant (3n-2)A^3\frac{|\tau^2|}{2!} \equiv \alpha_2|\tau^2|/2! ,$$

$$\left|x_{io}^{(3)}\frac{\tau^3}{3!}\right| \leqslant (3n-2)3\Theta A^{\mu+2}\frac{|\tau^3|}{3!} \equiv \alpha_3|\tau^3|/3! ,$$

$$\vdots$$

$$\left|x_{io}^{(\nu)}\frac{\tau^\nu}{\nu!}\right| \leqslant (3n-2)3\Theta^{\nu-2}(\mu+2)\cdots\cdots(\nu(\mu-1)-3\mu+6)A^{\nu(\mu-1)-2\mu+5} \equiv \alpha_\nu\frac{|\tau^\nu|}{\nu!}$$

$$\vdots \qquad (i=1,\dots n) \quad \tau\in[-\tau_1,\tau_1] .$$

We form the series

$$\sum_{\nu=2}^{\infty}\alpha_\nu\frac{|\tau^\nu|}{\nu!} , \tag{A-1.9}$$

and we take the limit of the ratio of two successive terms of it

$$\lim_{\nu\to\infty}\frac{\alpha_{\nu+1}\dfrac{|\tau^{\nu+1}|}{(\nu+1)!}}{\alpha_\nu\dfrac{|\tau^\nu|}{\nu!}} = |\tau|\lim_{\nu\to\infty}\frac{\alpha_{\nu+1}}{\alpha_\nu(\nu+1)} =$$

$$= |\tau|\lim_{\nu\to\infty}\frac{(3n-2)3\Theta^{\nu-1}(\mu+2)(2\mu+1)\cdots\cdots(\nu(\mu-1)-3\mu+6)(\nu(\mu-1)-2\mu+5)A^{\nu(\mu-1)-\mu+4}}{(\nu+1)(3n-2)3\Theta^{\nu-2}(\mu+2)(2\mu+1)\cdots\cdots(\nu(\mu-1)-3\mu+6)A^{\nu(\mu-1)-2\mu+5}} =$$

$$= |\tau|\lim_{\nu\to\infty}\frac{\Theta(\nu(\mu-1)-2\mu+5)A^{\mu-1}}{(\nu+1)} =$$

$$|\tau|\,\theta A^{\mu-1}\left[\lim_{\nu\to\infty}\frac{\nu(\mu-1)}{\nu+1}+\lim_{\nu\to\infty}\frac{-2\mu+5}{\nu+1}\right]=$$

$$=|\tau|\,\theta A^{\mu-1}(\mu-1)\,.$$

We can choose the interval $[-\tau_1\,,\,\tau_1]$ such that

$$|\tau|\,\theta A^{\mu-1}(\mu-1)<1\,.$$

In this interval the series (A-1.9) converges (ratio test). Hence the convergence of the series

$$\sum_{\nu=0}^{\infty}\left|x_{io}^{(\nu)}\frac{\tau^{\nu}}{\nu!}\right| \qquad (i=1,\ldots n)\quad \tau\in[-\tau_1\,,\tau_1]\,, \tag{A-1.10}$$

follows immediately from the relations (A-1.8). We interpret $x_{io}^{(0)}$ to mean x_{io}.

The convergence of the series

$$\sum_{\nu=0}^{\infty}x_{io}^{(\nu)}\frac{\tau^{\nu}}{\nu!} \qquad (i=1,\ \ldots n)\,, \tag{A-.1.11}$$

in the interval $[-\tau_1\,,\,\tau_1]$ follows as a result of the convergence of the series (A-1.10)

With a similar process we can prove that the series

$$\sum_{\nu=0}^{\infty}y_{io}^{(\nu)}\frac{\tau^{\nu}}{\nu!} \qquad (i=1,\ldots n)\,, \tag{A-1.12}$$

$$\sum_{\nu=0}^{\infty}z_{io}^{(\nu)}\frac{\tau^{\nu}}{\nu!} \qquad (i=1,\ \ldots n)\,, \tag{A-1.13}$$

converge.

The common interval of convergence of the series (A-1.11), (A-1.12) and (A-1.13) is the interval of convergence of the series (3.37).

APPENDIX A-2

Determination of the mass of the asteroid

Use of the total energy of the system to test the accuracy of the numerical applications

The problem which arises in the numerical application of the power series method concerns the determination of the mass of the asteroid.

A first estimation can be obtained by using the relation

$$\log p = 5.94 - 2\log R - 0.4\,B(1,0)$$

where p is the albedo factor, with probable value 0.16, R the radius of the asteroid in Km and $B(1,0)$ its absolute magnitude, given in the Ephemeris for the Minor Planets.

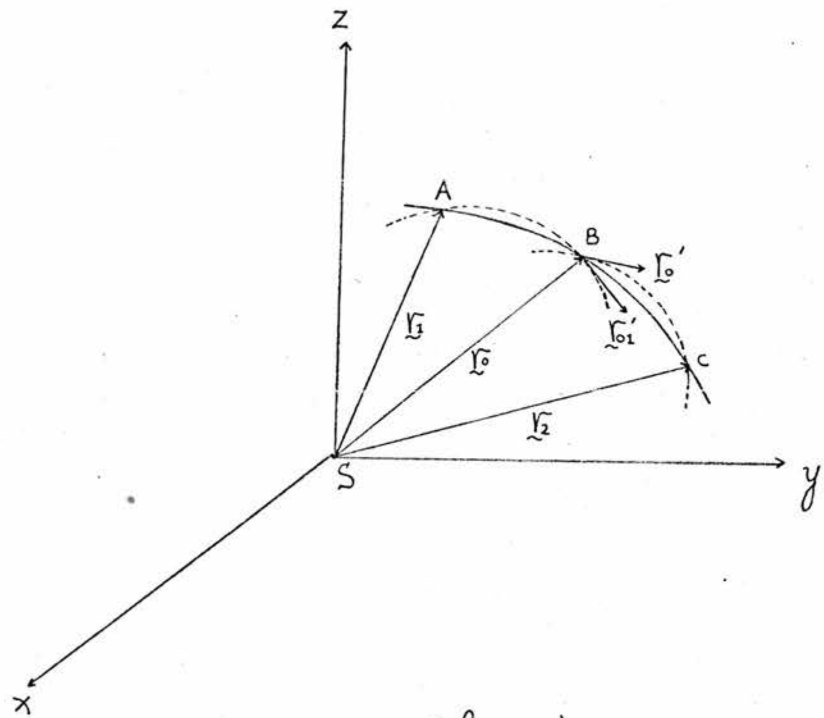
With probable density 3,5 $gr\ cm^{-3}$, and considering the shape of the asteroid as spherical, a value equal to $0.97\ 10^{-13}$ solar masses, is obtained for the mass of the asteroid.

Although this value is satisfactory, because the mass of the asteroid is very small compared with the masses of the planets, we calculated the mass of the asteroid as follows:

Consider three points on the real orbit of the asteroid A, B, C at three instants t_1, t_0, t_2 and the position vectors $\underset{\sim}{r_1}, \underset{\sim}{r_0}, \underset{\sim}{r_2}$. The use of the relation (3.44) twice for the values $\underset{\sim}{r_1}, \underset{\sim}{r_0}, t_1, t_0$ and $\underset{\sim}{r_0}, \underset{\sim}{r_2}, t_0, t_2$ gives two velocity vectors $\underset{\sim}{r_0}'$ and $\underset{\sim}{r_{01}}'$ at the point B, which must be the same.

The path of the asteroid in space depends on its mass, hence values of the mass different from the true value will give either a more curved or less curved path (fig. 1), hence the two velocity vectors $\underset{\sim}{r_0}'$ and $\underset{\sim}{r_{01}}'$ will not be the same.

The mass which gives the same values for the vectors at point B is the true mass of the asteroid .



(fig. 1)

This method gives a mass equal to 1.10^{-13} solar masses.

Another problem which arises in the power series method concerns the checks during the calculation. Here, the total energy of the system was used as a check.

We notice that the velocity of the Sun (origin of the co-ordinate system) is very nearly constant since the centre of mass of the system is inside the Sun, hence the value of the total energy must be very nearly constant.

Another test of the accuracy of the calculations, used in the numerical application, is provided by the mass of the minor planet calculated from the solution of the equations of motion. This should, of course, be constant.

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